

POLYNOMIALITY FOR THE POISSON CENTRE OF TRUNCATED MAXIMAL PARABOLIC SUBALGEBRAS.

FLORENCE FAUQUANT-MILLET AND POLYXENI LAMPROU

ABSTRACT. We show that the Poisson centre of truncated maximal parabolic subalgebras of a simple Lie algebra of type B, D and E_6 is a polynomial algebra.

In roughly half of the cases the polynomiality of the Poisson centre was already known by a completely different method.

For the rest of the cases, our approach is to construct an *algebraic slice* in the sense of Kostant given by an *adapted pair* and the computation of an improved upper bound for the Poisson centre.

1. INTRODUCTION

The base field k is assumed to be algebraically closed of characteristic zero.

In this paper we continue our study on the Poisson semicentre of maximal parabolic subalgebras of a simple Lie algebra over k , that we initiated in [8].

Let \mathfrak{p} be a parabolic subalgebra of a simple Lie algebra \mathfrak{g} over k . Recall that the Poisson semicentre $Sy(\mathfrak{p})$ of \mathfrak{p} is the vector space generated by the semi-invariants of the symmetric algebra $S(\mathfrak{p})$ of \mathfrak{p} and that the canonical truncation of \mathfrak{p} , denoted by \mathfrak{p}_Λ , is the largest subalgebra of \mathfrak{p} which vanishes on the weights of $Sy(\mathfrak{p})$. Hence one has that $Sy(\mathfrak{p}_\Lambda) = Y(\mathfrak{p}_\Lambda) := S(\mathfrak{p}_\Lambda)^{\mathfrak{p}_\Lambda}$. Observe that the latter, which is the algebra of invariants of $S(\mathfrak{p}_\Lambda)$ under the adjoint action of \mathfrak{p}_Λ , is also the Poisson centre of $S(\mathfrak{p}_\Lambda)$ (of \mathfrak{p}_Λ for short), where $S(\mathfrak{p}_\Lambda)$ is equipped with its natural Poisson structure. Since \mathfrak{p} is algebraic, one has that $Sy(\mathfrak{p}) = Y(\mathfrak{p}_\Lambda)$ (see 2.3 for more details). Observe also that the Poisson centre $Y(\mathfrak{p})$ of \mathfrak{p} is reduced to k , when \mathfrak{p} is not equal to \mathfrak{g} (see for example [12, 7.9] or [4, Chap. I, Sec. B, 8.2 (iv)]), whereas the Poisson semicentre $Sy(\mathfrak{p})$ of \mathfrak{p} is never reduced to scalars by [2].

By [5] - see also [12], [13] in the more general case of biparabolic (seaweed) subalgebras - we know that $Sy(\mathfrak{p})$ is lower and upper bounded, up to gradations, by polynomial algebras \mathcal{A} and \mathcal{B} respectively, having the same number of variables and whose weights may either be equal or differ by one half. Moreover, it was shown that the coincidence of these bounds is a sufficient condition for the polynomiality of

Key words and phrases. Poisson centre, (bi)parabolic subalgebras, polynomiality, adapted pairs.
AMS Classification : 17B35, 17B22, 16W22.

$Sy(\mathfrak{p})$. The coincidence of these bounds occurs often, for instance when \mathfrak{g} is simple of type A or C and \mathfrak{p} is any parabolic subalgebra of \mathfrak{g} .

However, the coincidence of bounds \mathcal{A} and \mathcal{B} is not a necessary condition for the polynomiality of the Poisson semicentre and indeed there are examples where \mathcal{A} and \mathcal{B} do not coincide but the Poisson semicentre is polynomial, for example in the Borel case [11].

Since $Sy(\mathfrak{p}_\Lambda) = Y(\mathfrak{p}_\Lambda)$, the field of invariant fractions $C(\mathfrak{p}_\Lambda) := (\text{Fract } S(\mathfrak{p}_\Lambda))^{\mathfrak{p}_\Lambda}$ is equal to the field of fractions $\text{Fract } (Y(\mathfrak{p}_\Lambda))$ of $Y(\mathfrak{p}_\Lambda)$, as each semi-invariant of $\text{Fract } S(\mathfrak{a})$ is a quotient of two semi-invariants of $S(\mathfrak{a})$, for any finite dimensional Lie algebra \mathfrak{a} by [2] or [4, Chap. I, Sec. B, 5.11, 5.12]. Hence the polynomiality of $Sy(\mathfrak{p}) = Y(\mathfrak{p}_\Lambda)$ implies that the field of invariant fractions $C(\mathfrak{p}_\Lambda)$ is a purely transcendental extension of the base field k and by [20, Thm. 66] so is the field of invariant fractions $C(\mathfrak{p})$, since there exists a set of algebraically independent generators of $Sy(\mathfrak{p})$ formed by weight vectors, that is by semi-invariants of $S(\mathfrak{p})$. This allows us to answer positively Dixmier's fourth problem for such parabolic subalgebras, namely whether the field of invariant fractions is a purely transcendental extension of the base field, for any finite dimensional Lie algebra. However the polynomiality of the Poisson centre $Y(\mathfrak{p}_\Lambda)$ is a much stronger result.

Recently, several authors have been interested in the question of polynomiality of the Poisson centre of non-reductive algebraic Lie algebras; parabolic and biparabolic (seaweed) subalgebras of a simple Lie algebra \mathfrak{g} over k were studied in [5], [6], [12], [13] and some particular semi-direct products were studied in [22], [23], [24], [27], [28], where polynomiality of the Poisson centre was shown. In [20] the author gives necessary and sufficient conditions for the Poisson centre or semicentre of certain finite dimensional Lie algebra to be polynomial.

So far, only one counterexample to the polynomiality of the Poisson semicentre of a biparabolic subalgebra \mathfrak{p} is known, namely when \mathfrak{g} is of type E_8 and \mathfrak{p} is the maximal parabolic subalgebra of \mathfrak{g} , whose canonical truncation coincides with the centralizer of the highest root vector of \mathfrak{g} [26].

In [8] we studied $Sy(\mathfrak{p})$ for \mathfrak{p} a maximal parabolic subalgebra of a simple Lie algebra \mathfrak{g} , when the lower and upper bounds \mathcal{A} and \mathcal{B} coincide (hence $Sy(\mathfrak{p})$ is polynomial) and we constructed slices for the coadjoint action, extending the Kostant Slice Theorem [19, Thm. 0.10].

In this paper we study the remaining cases for \mathfrak{g} simple of type B, D and E_6 and we deduce the polynomiality of the Poisson semicentre $Sy(\mathfrak{p})$ by constructing slices for the coadjoint action and computing an “improved upper bound” (see below).

The slices we constructed in [8] were given by *adapted pairs* (see 2.8) for the canonical truncations \mathfrak{p}_Λ of the parabolic subalgebras \mathfrak{p} that we studied. In this paper we construct adapted pairs for the remaining cases mentioned above.

Adapted pairs play the role of principal \mathfrak{sl}_2 -triples in the non-reductive case and were introduced in [14]. They give an improved upper bound \mathcal{B}' for the character of $Sy(\mathfrak{p}) = Y(\mathfrak{p}_\Lambda)$ [16]. When this bound is attained, in particular when it coincides with the character of the lower bound \mathcal{A} mentioned above, polynomiality of $Sy(\mathfrak{p})$ follows and the adapted pair gives an algebraic slice (in the sense of [17, 7.6]) also called a Weierstrass section in [7], extending the Kostant Slice Theorem [19, Thm. 0.10] to non-reductive Lie algebras. By [7], this Weierstrass section is also an affine slice for the coadjoint action (in the sense of [17, 7.3]).

Some particular cases had already been studied by other authors and different methods. For example, it was shown in [21] that for all maximal parabolic subalgebras \mathfrak{p} whose canonical truncation is the centralizer of the highest root vector of the simple Lie algebra (except in type E_8 , where we have Yakimova's counterexample), the Poisson semicentre $Sy(\mathfrak{p})$ is a polynomial algebra over k .

Furthermore, Heckenberger [10] showed by computer calculations that in type B_n , $2 \leq n \leq 4$, the Poisson semicentre $Sy(\mathfrak{p})$ is polynomial for all parabolic subalgebras \mathfrak{p} .

In [25] an affine slice for the coadjoint action of \mathfrak{p} was constructed for some non truncated biparabolic subalgebras \mathfrak{p} of a simple Lie algebra, which gave a positive answer to Dixmier's fourth problem for $C(\mathfrak{p})$. These biparabolic subalgebras \mathfrak{p} do not coincide with the maximal parabolic subalgebras we are interested in.

Below, labeling of simple roots follows Bourbaki [1, Planches I-IX].

Adapted pairs need not exist for all truncated parabolic subalgebras and are very hard to construct in general. One may hope to construct such pairs when the truncated Cartan subalgebra - that is, the subalgebra of the Cartan subalgebra, which is contained in the canonical truncation of the parabolic subalgebra we consider - is large enough, as it happens when \mathfrak{g} is of type A or when the parabolic subalgebra \mathfrak{p} is maximal; however, we showed that even in these favourable cases adapted pairs may not exist, as it happens for example when \mathfrak{g} is of type F_4 and \mathfrak{p} is the maximal parabolic subalgebra corresponding to $\pi' = \{\alpha_1, \alpha_2, \alpha_4\}$ [8, Sect. 10]. In type A adapted pairs were constructed for all truncated biparabolic subalgebras in [15].

When the parabolic subalgebra \mathfrak{p} is maximal associated to $\pi' = \pi \setminus \{\alpha_s\}$ where π is a set of simple roots α_i , $1 \leq i \leq n$, in \mathfrak{g} and \mathfrak{g} is simple of type B_n or D_n , the bounds \mathcal{A} and \mathcal{B} for $Sy(\mathfrak{p})$ coincide exactly when s is odd (in type D_n , $n \geq 4$, under the restriction $s \neq n-1$; additionally, when $s = n-1$ and s even, and finally in type D_4 for all s except for $s = 2$; in type B_n , $n \geq 2$, also for $n = s = 2$ and $n = s = 4$).

In this paper we give an adapted pair for the rest of the truncated maximal parabolic subalgebras in type B and D. In particular, we prove a lemma of non-degeneracy (3.8) which is a non-obvious generalization of [8, Lemma 5].

Using GAP [9], we also construct an adapted pair for the truncated maximal parabolic subalgebras \mathfrak{p}_Λ in a simple Lie algebra of type E_6 , when the bounds \mathcal{A} and

\mathcal{B} do not coincide, that is when $s = 1, 6$ (for $s = 2$ an adapted pair was already constructed in [16]).

From the case D_6 , $s = 6$, we also deduce an adapted pair for the truncated maximal parabolic subalgebra of \mathfrak{g} of type E_7 corresponding to $\pi' = \pi \setminus \{\alpha_3\}$ (6.7).

We compute the improved upper bound \mathcal{B}' (4.6.2, 5.6.2, 6.6.2, 6.7, 7) and we show that it is attained and hence the Poisson centre $Y(\mathfrak{p}_\Lambda)$ of \mathfrak{p}_Λ is polynomial (4.6.3, 5.6.3, 6.6.3, 6.7, 7). We deduce that for all such maximal parabolic subalgebras \mathfrak{p} , Dixmier's fourth problem is true for $C(\mathfrak{p})$. Furthermore, as in [8] we obtain an algebraic and an affine slice for the dual of \mathfrak{p}_Λ .

Acknowledgements. This work was initiated when the second author was visiting the University Jean Monnet, Saint-Etienne. We would like to thank A. Joseph for many fruitful discussions on adapted pairs and for his interest in our work. We are also grateful to A. Ooms for enlightening exchange of ideas on the polynomiality of the Poisson semicentre. Part of these results were presented in the seminar at the Weizmann Institute of Science in Israel in April 2016 and in the conference "Representation Theory in Samos" in Greece in July 2016.

2. PRELIMINARIES.

2.1. Let \mathfrak{g} be a finite dimensional simple Lie algebra over k and \mathfrak{h} a fixed Cartan subalgebra of \mathfrak{g} .

Let Δ be the root system of \mathfrak{g} with respect to \mathfrak{h} , π a chosen set of simple roots, Δ^+ (resp. Δ^-) the set of positive (resp. negative) roots. We adopt the labeling of [1, Planches I-IX] for the simple roots in π .

For any $\alpha \in \Delta$, let \mathfrak{g}_α denote the corresponding root space of \mathfrak{g} and fix a nonzero vector x_α in \mathfrak{g}_α . Then $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^-$, where $\mathfrak{n} = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$ and $\mathfrak{n}^- = \bigoplus_{\alpha \in \Delta^-} \mathfrak{g}_\alpha$. For all $\alpha \in \pi$, denote by α^\vee the corresponding coroot. For any subset A of Δ , set $\mathfrak{g}_A = \bigoplus_{\alpha \in A} \mathfrak{g}_\alpha$.

2.2. For any subset π' of π , let $\Delta_{\pi'}$ be the subset of roots in Δ generated by π' and $\Delta_{\pi'}^+$, $\Delta_{\pi'}^-$ the sets of positive and negative roots in $\Delta_{\pi'}$ respectively.

One defines the standard parabolic subalgebra $\mathfrak{p}_{\pi'}$ associated to π' to be the algebra $\mathfrak{p}_{\pi'} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}_{\pi'}^-$ where $\mathfrak{n}_{\pi'}^- = \bigoplus_{\alpha \in \Delta_{\pi'}^-} \mathfrak{g}_\alpha$. Its opposed algebra then is $\mathfrak{p}_{\pi'}^- = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}_{\pi'}^+$, with $\mathfrak{n}_{\pi'}^+$ defined similarly. The dual space $\mathfrak{p}_{\pi'}^*$ identifies with $\mathfrak{p}_{\pi'}^-$ via the Killing form K on \mathfrak{g} .

We denote by $W_{\pi'}$ the Weyl group associated to π' and by r_γ , for $\gamma \in \Delta_{\pi'}$ the reflection with respect to γ . Then $W_{\pi'}$ is the subgroup of the Weyl group of $(\mathfrak{g}, \mathfrak{h})$ generated by r_γ , for all $\gamma \in \Delta_{\pi'}$.

2.3. Let \mathfrak{a} be a finite dimensional Lie algebra over k . The Poisson semicentre $Sy(\mathfrak{a})$ of its symmetric algebra $S(\mathfrak{a})$ (of \mathfrak{a} for short) is defined to be the vector space spanned

by the semi-invariants under the adjoint action of \mathfrak{a} that is, $Sy(\mathfrak{a}) = \bigoplus_{\lambda \in \mathfrak{a}^*} S(\mathfrak{a})_\lambda$ where $S(\mathfrak{a})_\lambda = \{s \in S(\mathfrak{a}) \mid \forall x \in \mathfrak{a}, (\text{ad } x)s = \lambda(x)s\}$. It is a subalgebra of $S(\mathfrak{a})$. When $S(\mathfrak{a})_\lambda \neq \{0\}$, λ is called a weight of the Poisson semicentre $Sy(\mathfrak{a})$. Let $\Lambda(\mathfrak{a})$ denote the set of weights of $Sy(\mathfrak{a})$. Then the canonical truncation \mathfrak{a}_Λ of \mathfrak{a} is : $\mathfrak{a}_\Lambda = \bigcap_{\lambda \in \Lambda(\mathfrak{a})} \ker \lambda$. It is an ideal of \mathfrak{a} containing the derived subalgebra of \mathfrak{a} .

Equip $S(\mathfrak{a})$ with its natural Poisson structure coming from the Lie bracket on \mathfrak{a} . The Poisson centre $Y(\mathfrak{a})$ of \mathfrak{a} is the centre of $S(\mathfrak{a})$ for this structure and it is also the set of the invariants in $S(\mathfrak{a})$ under the adjoint action of \mathfrak{a} , that is $Y(\mathfrak{a}) = S(\mathfrak{a})_0$. It is an algebra contained in the Poisson semicentre $Sy(\mathfrak{a})$ of $S(\mathfrak{a})$.

If \mathfrak{a} is almost algebraic (for a definition see [6, B.2]), then we have that $Sy(\mathfrak{a}) = Sy(\mathfrak{a}_\Lambda) = Y(\mathfrak{a}_\Lambda)$ (see [6, B.2] or [20, Thm. 4]).

2.4. The index of \mathfrak{a} , denoted by $\text{ind } \mathfrak{a}$, is the minimal dimension of a stabilizer \mathfrak{a}^f for $f \in \mathfrak{a}^*$. When \mathfrak{a} is algebraic the index of \mathfrak{a} is also equal to the minimal codimension of a coadjoint orbit in \mathfrak{a}^* [3, 1.11.3].

An element $y \in \mathfrak{a}^*$ is called regular in \mathfrak{a}^* if its stabilizer \mathfrak{a}^y is of minimal dimension (equal to $\text{ind } \mathfrak{a}$).

2.5. Let $\pi' \subset \pi$ and $\mathfrak{p}_{\pi', \Lambda}$ be the canonical truncation of $\mathfrak{p}_{\pi'}$ which we recall is defined to be the largest subalgebra of $\mathfrak{p}_{\pi'}$ that vanishes on the weights of $Sy(\mathfrak{p}_{\pi'})$. It has the property that the Poisson centre $Y(\mathfrak{p}_{\pi', \Lambda})$ is equal to the Poisson semicentre $Sy(\mathfrak{p}_{\pi', \Lambda})$ and also equal to $Sy(\mathfrak{p}_{\pi'})$, since $\mathfrak{p}_{\pi'}$ is algebraic and so also almost algebraic.

The canonical truncation of $\mathfrak{p}_{\pi'}$ was given explicitly in [6]. It is of the form $\mathfrak{p}_{\pi', \Lambda} = \mathfrak{n} \oplus \mathfrak{h}_\Lambda \oplus \mathfrak{n}_{\pi'}^-$ where \mathfrak{h}_Λ is a subalgebra of \mathfrak{h} called the truncated Cartan subalgebra (this is the largest subalgebra of \mathfrak{h} which vanishes on the set of weights of all semi-invariants in $Sy(\mathfrak{p}_{\pi'})$).

The Gelfand-Kirillov dimension of $Y(\mathfrak{p}_{\pi', \Lambda})$ is equal to the index of $\mathfrak{p}_{\pi', \Lambda}$. For more details, see [6, 2.4, 2.5, B.2].

Let $\mathfrak{h}' \subset \mathfrak{h}$ be the Cartan subalgebra of the Levi factor of $\mathfrak{p}_{\pi'}$. When $\pi' = \pi \setminus \{\alpha_s\}$, then $\mathfrak{h}_\Lambda = \mathfrak{h}'$ that is, \mathfrak{h}_Λ is the vector space over k generated by all α^\vee with $\alpha \in \pi'$.

For convenience, we replace the truncated parabolic subalgebra $\mathfrak{p}_{\pi', \Lambda}$ by its opposed algebra $\mathfrak{p}_{\pi', \Lambda}^-$ (that is, the canonical truncation of the opposed algebra $\mathfrak{p}_{\pi'}^-$). From now on, we denote it simply by \mathfrak{p} .

2.6. For any \mathfrak{h} -module $M = \bigoplus_{\nu \in \mathfrak{h}^*} M_\nu$ with finite dimensional weight spaces $M_\nu := \{m \in M \mid \forall h \in \mathfrak{h}, h.m = \nu(h)m\}$, we may define its formal character by

$$\text{ch } M = \sum_{\nu \in \mathfrak{h}^*} \dim M_\nu e^\nu.$$

Given two such \mathfrak{h} -modules M and M' write $\text{ch } M \leq \text{ch } M'$ if $\dim M_\nu \leq \dim M'_\nu$ for all $\nu \in \mathfrak{h}^*$ [13, 2.8].

2.7. Here we recall the formal characters $\text{ch } \mathcal{A}$ and $\text{ch } \mathcal{B}$ of the lower and the upper bounds mentioned in the introduction for $\text{ch } Y(\mathfrak{p})$ given in [13, Thm. 6.7].

Let $E(\pi')$ be the set of $\langle \mathbf{i}\mathbf{j} \rangle$ -orbits of π , where \mathbf{i} and \mathbf{j} are the involutions of π defined for example in [8, 2.2]. Denote by $\{\varpi_\alpha\}_{\alpha \in \pi}$ (resp. $\{\varpi'_\alpha\}_{\alpha \in \pi'}$) the set of fundamental weights associated to π (resp. to π'); the same sets sometimes are denoted by $\{\varpi_i\}_{\alpha_i \in \pi}$ and $\{\varpi'_i\}_{\alpha_i \in \pi'}$ respectively. Let \mathcal{B}_π (resp. $\mathcal{B}_{\pi'}$) be the set of weights of the Poisson semicentre of $S(\mathfrak{n} \oplus \mathfrak{h})$ (resp. $S(\mathfrak{n}_{\pi'} \oplus \mathfrak{h}')$).

For all $\Gamma \in E(\pi')$, set

$$\delta_\Gamma = - \sum_{\gamma \in \Gamma} \varpi_\gamma - \sum_{\gamma \in \mathbf{j}(\Gamma)} \varpi_\gamma + \sum_{\gamma \in \Gamma \cap \pi'} \varpi'_\gamma + \sum_{\gamma \in \mathbf{i}(\Gamma \cap \pi')} \varpi'_\gamma$$

and

$$\varepsilon_\Gamma = \begin{cases} 1/2 & \text{if } \Gamma = \mathbf{j}(\Gamma), \text{ and } \sum_{\gamma \in \Gamma} \varpi_\gamma \in \mathcal{B}_\pi, \text{ and } \sum_{\gamma \in \Gamma \cap \pi'} \varpi'_\gamma \in \mathcal{B}_{\pi'}. \\ 1 & \text{otherwise.} \end{cases}$$

It is shown in [13, Thm. 6.7] that

$$\prod_{\Gamma \in E(\pi')} (1 - e^{\delta_\Gamma})^{-1} \leq \text{ch } Y(\mathfrak{p}) \leq \prod_{\Gamma \in E(\pi')} (1 - e^{\varepsilon_\Gamma \delta_\Gamma})^{-1}.$$

In particular, if for all $\Gamma \in E(\pi')$, $\varepsilon_\Gamma = 1$, the above inequalities are equalities and $Y(\mathfrak{p})$ is a polynomial algebra over k [5].

2.8. An adapted pair for \mathfrak{p} is a pair $(h, y) \in \mathfrak{h}_\Lambda \times \mathfrak{p}^*$ such that y is regular in \mathfrak{p}^* , and $(\text{ad } h)y = -y$ where ad denotes the coadjoint action of \mathfrak{p} on \mathfrak{p}^* .

2.9. Assume that there exists an adapted pair $(h, y) \in \mathfrak{h}_\Lambda \times \mathfrak{p}^*$ for \mathfrak{p} . One may choose subsets $S, T \subset \Delta^+ \sqcup \Delta_\pi^-$ such that $y = \sum_{\gamma \in S} a_\gamma x_\gamma$, with $a_\gamma \in k \setminus \{0\}$ for all $\gamma \in S$, and $\mathfrak{p}^* = (\text{ad } \mathfrak{p})y \oplus \mathfrak{g}_T$. Note that we may choose T such that $|T| = \text{ind } \mathfrak{p}$. Assume further that $S|_{\mathfrak{h}_\Lambda}$ is a basis for \mathfrak{h}_Λ^* . Then for each $\gamma \in T$ there exists a unique $t(\gamma) \in \mathbb{Q}S$ such that $\gamma + t(\gamma)$ vanishes on \mathfrak{h}_Λ .

By [16, Lem. 6.11]

$$\text{ch } Y(\mathfrak{p}) \leq \prod_{\gamma \in T} (1 - e^{-(\gamma + t(\gamma))})^{-1}$$

and we will call the right hand side an “improved upper bound” for $\text{ch } Y(\mathfrak{p})$; in this work it is indeed always an improvement of the upper bound mentioned in 2.7.

Moreover by [16, Lem. 6.11] if the lower bound in 2.7 and this improved upper bound coincide then the restriction map gives an isomorphism of algebras $Y(\mathfrak{p}) \simeq R[y + \mathfrak{g}_T]$, where $R[y + \mathfrak{g}_T]$ is the ring of polynomial functions on $y + \mathfrak{g}_T$, isomorphic to $S(\mathfrak{g}_T^*)$. Hence $Y(\mathfrak{p})$ is a polynomial algebra over k and $y + \mathfrak{g}_T$ is an algebraic slice in the sense of [17, 7.6], also called a Weierstrass section in [7] and by [7] it is also

an affine slice in the sense of [17, 7.3] for the coadjoint action of the adjoint group of \mathfrak{p} on \mathfrak{p}^* .

2.10. Assume that there exists an adapted pair (h, y) for \mathfrak{p} and denote by V an h -stable complement of $(\text{ad } \mathfrak{p})y$ in \mathfrak{p}^* . Assume further that $Y(\mathfrak{p})$ is a polynomial algebra and let f_1, \dots, f_l be homogeneous generators for $Y(\mathfrak{p})$ ($l = \text{ind } \mathfrak{p}$). Then by [18, Cor. 2.3] if m_1, \dots, m_l are the eigenvalues of h on an h -stable basis of V , one has that $\deg f_i = m_i + 1$ for all $1 \leq i \leq l$, up to a permutation of indices.

3. A LEMMA OF REGULARITY

Keep the notations of the previous section and let $f \in \mathfrak{g}$ and $\Phi_f : \mathfrak{g} \times \mathfrak{g} \rightarrow k$ be the skew-symmetric bilinear form on \mathfrak{g} defined by $\Phi_f(x, x') = K(f, [x, x'])$. Here we recall ([8, Def. 2]) the definition of an Heisenberg set, of centre $\gamma \in \Delta$. It is a subset Γ_γ of Δ such that $\gamma \in \Gamma_\gamma$ and for all $\alpha \in \Gamma_\gamma \setminus \{\gamma\}$, there exists a (unique) $\alpha' \in \Gamma_\gamma \setminus \{\gamma\}$ such that $\alpha + \alpha' = \gamma$.

Let $\mathfrak{p}_{\pi', \Lambda} = \mathfrak{n} \oplus \mathfrak{h}_\Lambda \oplus \mathfrak{n}_{\pi'}^-$ be the truncated parabolic subalgebra of \mathfrak{g} associated to $\pi' \subset \pi$. Let S be a subset of $\Delta^+ \sqcup \Delta_{\pi'}^-$ and for all $\gamma \in S$ choose an Heisenberg set Γ_γ of centre γ in $\Delta^+ \sqcup \Delta_{\pi'}^-$. Assume that the sets Γ_γ are disjoint and set $\Gamma = \bigsqcup_{\gamma \in S} \Gamma_\gamma$ and $y = \sum_{\gamma \in S} a_\gamma x_\gamma$, where $a_\gamma \in k \setminus \{0\}$ for all $\gamma \in S$. Set $O = \bigsqcup_{\gamma \in S} \Gamma_\gamma^0$, with $\Gamma_\gamma^0 = \Gamma_\gamma \setminus \{\gamma\}$, and $\mathfrak{o} = \mathfrak{g}_{-O}$.

3.1. The lemma below follows exactly like [8, Lem. 6].

Lemma. *Retain the above notations and hypotheses and assume further that*

- (i) *The restriction of Φ_y to $\mathfrak{o} \times \mathfrak{o}$ is non-degenerate.*
- (ii) *$S|_{\mathfrak{h}_\Lambda}$ is a basis for \mathfrak{h}_Λ^* .*
- (iii) *$|T| = \text{ind } \mathfrak{p}_{\pi', \Lambda}$, where $T = (\Delta^+ \sqcup \Delta_{\pi'}^-) \setminus \Gamma$.*

Then $\mathfrak{p}_{\pi', \Lambda} = (\text{ad } \mathfrak{p}_{\pi', \Lambda}^-)y \oplus \mathfrak{g}_T$, where ad denotes the coadjoint action. In particular, y is regular in $\mathfrak{p}_{\pi', \Lambda}$. Moreover, if we uniquely define $h \in \mathfrak{h}_\Lambda$ by the relations $\gamma(h) = -1$ for all $\gamma \in S$, then (h, y) is an adapted pair for $\mathfrak{p}_{\pi', \Lambda}^-$.

3.2. Given $\gamma \in S$, for all $\alpha \in \Gamma_\gamma^0$ denote by α' the unique root in Γ_γ^0 such that $\alpha + \alpha' = \gamma$ and let θ_γ be the involution in Γ_γ^0 mapping $\alpha \in \Gamma_\gamma^0$ to α' . Denote by θ the involution in O induced by all θ_γ , $\gamma \in S$.

Clearly, the non-degeneracy of the restriction of Φ_y to $\mathfrak{o} \times \mathfrak{o}$ is immediate if, for all $\alpha \in O$, the only root β in O such that $\alpha + \beta \in S$ is $\beta = \theta(\alpha)$.

Unfortunately this will not be the case in general but the lemma in 3.8 will give sufficient conditions for the non-degeneracy of the restriction of Φ_y to $\mathfrak{o} \times \mathfrak{o}$. To state this lemma, we need further notations. In particular for each root $\alpha \in O$, we set $S_\alpha = \{\beta \in O \mid \alpha + \beta \in S\}$ and for all $n \geq 1$, $O_n = \{\alpha \in O \mid |S_\alpha| = n\}$. Note that $O_1 = \{\alpha \in O \mid \forall \beta \in O, \alpha + \beta \in S \implies \beta = \theta(\alpha)\}$.

3.3. Sequences of roots. Keep the notations of 3.2 above. Let $\alpha \in O$. Set $\alpha^0 = \alpha$ and for all $i \in \mathbb{N}$ define $\alpha^i \in O$ inductively as follows. If $\theta(\alpha^i) \in O_1$, set $\alpha^{i+1} = \alpha^i$. Otherwise, let $\alpha^{i+1} \neq \alpha^i$ be a root in O such that $\alpha^{i+1} + \theta(\alpha^i) \in S$. For all $i \in \mathbb{N}$, set $\alpha^{(i)} = \theta(\alpha)^i$.

We will say that $(\alpha^i)_{i \in \mathbb{N}}$ is a sequence of roots in O constructed from α ; such a sequence always exists but in general is not unique. If for all $i \in \mathbb{N}$, $\theta(\alpha^i) \in O_1 \sqcup O_2$, then $(\alpha^i)_{i \in \mathbb{N}}$ will be called *the* sequence of roots in O constructed from α , since in this case, α^i is uniquely defined, for all $i \in \mathbb{N}$. Note that if $\theta(\alpha^i) \in O_1$ for some $i \in \mathbb{N}$, then $\alpha^j = \alpha^i$ for all $j \geq i$. Conversely, if $\alpha^i = \alpha^{i+1}$, then $\theta(\alpha^i) \in O_1$ and $\alpha^j = \alpha^i$, for all $j \geq i$. We call a minimal such i the rank of the sequence $(\alpha^j)_{j \in \mathbb{N}}$ and we say that the sequence is stationary at rank i . Note that if $\theta(\alpha^i) \notin O_1$ then $\alpha^{i+1} \notin O_1$.

3.4. Stability condition. Keep the notations of 3.2 and let $\alpha \in O$. We say that α satisfies the “stability condition” if there exists $\alpha' \in S_\alpha \setminus \{\theta(\alpha)\}$ such that $\alpha' \in O_2$ and $\theta(\alpha') \in O_1$. We say then that α' satisfies condition (St_α) .

In what follows (3.5), we will consider sequences $(\alpha^i)_{i \in \mathbb{N}}$ of roots in O constructed from α , such that $\alpha^i, \theta(\alpha^i) \in O_1 \sqcup O_2 \sqcup O_3$ for all $i \in \mathbb{N}$, and whenever one of them belongs to O_3 , then it satisfies the stability condition. First of all, for every $\beta \in O_3$ which satisfies the stability condition, we choose and fix a root $\beta' \in S_\beta \setminus \{\theta(\beta)\}$ which satisfies condition (St_β) . Now let $\alpha \in O$ and assume that we have defined the roots $\alpha^i \in O$ inductively as in 3.3, for all $0 \leq i \leq t$, $t \in \mathbb{N}$. Then if $\theta(\alpha^t) \in O_3$ satisfies the stability condition, the root α^{t+1} is defined to be the unique root in O distinct from α^t and distinct from the chosen root $\theta(\alpha^t)'$ satisfying condition $(St_{\theta(\alpha^t)})$ such that $\alpha^{t+1} + \theta(\alpha^t) \in S$; then $(\alpha^t)^1 = \alpha^{t+1}$. We proceed similarly for the sequence $(\alpha^{(i)})$.

Note that if $\theta(\alpha^i) \in O_1 \sqcup O_2 \sqcup O_3$, for all $i \in \mathbb{N}$ are such that whenever $\theta(\alpha^i) \in O_3$, it satisfies the stability condition, then the sequence $(\alpha^i)_{i \in \mathbb{N}}$ is uniquely defined.

3.5. Stationary condition. Let $\alpha \in O$ and set $A_\alpha = \{\alpha^i, \theta(\alpha^i) \mid i \in \mathbb{N}\}$ for a sequence $(\alpha^i)_{i \in \mathbb{N}}$ of roots in O constructed from α . We say that α satisfies the “stationary condition” if the three conditions below are satisfied:

- (i) $A_\alpha \subset O_1 \sqcup O_2 \sqcup O_3$.
- (ii) If $\beta \in A_\alpha \cap O_3$ then β satisfies the stability condition.
- (iii) The sequence $(\alpha^i)_{i \in \mathbb{N}}$ is stationary (and hence so is the sequence $(\theta(\alpha^i))_{i \in \mathbb{N}}$).

Remarks. (1) By 3.4, conditions (i) and (ii) for the set $\{\theta(\alpha^i) \mid i \in \mathbb{N}\}$ imply that the sequence $(\alpha^i)_{i \in \mathbb{N}}$ is uniquely defined.

(2) Suppose that $(\alpha^i)_{i \in \mathbb{N}}$ is a sequence of roots in O constructed from α and that, for $i_0 \in \mathbb{N}$, conditions (i) and (ii) are satisfied for the set $\{\theta(\alpha^i) \mid i \in \mathbb{N}, i \geq i_0\}$. Then for all $i, j \in \mathbb{N}$, $i \geq i_0$, one has that $(\alpha^i)^j = \alpha^{i+j}$ (it follows from the conventions in 3.4 and (1)).

(3) Assume that α satisfies the stationary condition. Then for all $i \in \mathbb{N}$, α^i satisfies the stationary condition. Conversely, assume that there exists $i_0 \in \mathbb{N}$ such that $A'_\alpha = \{\alpha^i, \theta(\alpha^i) \mid 0 \leq i \leq i_0 - 1\}$ satisfies (i) and (ii) and α^{i_0} satisfies the stationary condition. Then α satisfies the stationary condition. This easily follows by (2) above.

3.6. Stationary roots. We say that α is “a stationary root” and write $\alpha \in O_{st}$ if both α and $\theta(\alpha)$ satisfy the stationary condition. Clearly, if $\alpha \in O_{st}$, then $\theta(\alpha) \in O_{st}$.

Remarks. Let $\alpha \in O$.

(1) Assume that conditions (i) and (ii) of Section 3.5 are satisfied for the sets A_α and $A_{\theta(\alpha)}$ and that for all $i, s \in \mathbb{N}$, $0 \leq s \leq i - 1$, $\theta(\alpha^s), \alpha^{s+1} \in O_2$. Then for all $j \leq i$, $(\alpha^i)^{(j)} = \theta(\alpha^{i-j})$ and for all $j \geq i$, $(\alpha^i)^{(j)} = \alpha^{(j-i)}$.

(2) Assume that conditions (i) and (ii) of Section 3.5 are satisfied for the sets A_α and $A_{\theta(\alpha)}$ and that for all $i, s \in \mathbb{N}$, $0 \leq s \leq i - 1$, $\theta(\alpha^{(s)}), \alpha^{(s+1)} \in O_2$. Then for all $j \leq i$, $(\alpha^{(i)})^{(j)} = \theta(\alpha^{(i-j)})$ and for all $j \geq i$, $(\alpha^{(i)})^{(j)} = \alpha^{(j-i)}$.

(3) Assume that $A := A_\alpha \cup A_{\theta(\alpha)} = \{\alpha^i, \alpha^{(i)}, \theta(\alpha^i), \theta(\alpha^{(i)}) \mid i \in \mathbb{N}\} \subset O_1 \sqcup O_2$. If $\alpha \in O_{st}$ then $A \subset O_{st}$ and conversely if there exists $i_0 \in \mathbb{N}$ such that α^{i_0} or $\alpha^{(i_0)}$ belongs to O_{st} , then $\alpha \in O_{st}$. This easily follows from (1) and (2) above and from remark (2) in 3.5.

(4) Assume that there exists $i_0 \in \mathbb{N}$ such that $\alpha^{i_0} \in O_{st}$, $A'_\alpha = \{\alpha^i, \theta(\alpha^i) \mid 0 \leq i \leq i_0 - 1\} \subset O_1 \sqcup O_2 \sqcup O_3$ and if $\beta \in A'_\alpha \cap O_3$, then β satisfies the stability condition, and that $\theta(\alpha)$ satisfies the stationary condition. From remark (3) in 3.5 we deduce that $\alpha \in O_{st}$. We have a similar statement if we interchange α and $\theta(\alpha)$.

(5) Assume that there exists $i_0, i_1 \in \mathbb{N}$ such that $A'_\alpha = \{\alpha^i, \theta(\alpha^i) \mid 0 \leq i \leq i_0 - 1\} \subset O_1 \sqcup O_2 \sqcup O_3$, $A'_{\theta(\alpha)} = \{\alpha^{(i)}, \theta(\alpha^{(i)}) \mid 0 \leq i \leq i_1 - 1\} \subset O_1 \sqcup O_2 \sqcup O_3$, $\alpha^{i_0}, \alpha^{(i_1)} \in O_{st}$ and if $\beta \in (A'_\alpha \cup A'_{\theta(\alpha)}) \cap O_3$, then β satisfies the stability condition. Then from (4) above it follows that $\alpha \in O_{st}$.

(6) Assume that $\alpha \in O_3$ and satisfies the stationary condition. By (ii) of the definition in 3.5, α also satisfies the stability condition. Let $\alpha' \in S_\alpha \setminus \{\theta(\alpha)\}$ be the chosen root satisfying condition (St_α) . Then $\alpha' \in O_{st}$: indeed, since $\alpha' \in O_2 \cap S_\alpha$, one has that $(\alpha')^{(1)} = \theta(\alpha')^1 = \alpha$, then $\theta(\alpha')$ satisfies the stationary condition, because α does, by remark (3) of 3.5. Moreover $\theta(\alpha') \in O_1$ implies that $(\alpha')^1 = \alpha'$, then α' also satisfies the stationary condition.

Lemma. Let $\alpha \in O_{st}$ and set $A = \{\alpha^i, \theta(\alpha^i), \alpha^{(i)}, \theta(\alpha^{(i)}) \mid i \in \mathbb{N}\}$. Let $\vartheta : O \rightarrow O$ be a permutation such that, for all $\gamma \in O$, $\gamma + \vartheta(\gamma) \in S$. Then the restriction of ϑ on A coincides with θ and if $\beta \in A \cap O_3$, and $\beta' \in S_\beta \setminus \{\theta(\beta)\}$ is the chosen root satisfying condition (St_β) , then ϑ exchanges β' and $\theta(\beta')$.

Proof. Denote by n_0 (resp. n_1) the rank of the stationary sequence $(\alpha^i)_{i \in \mathbb{N}}$ (resp. $(\alpha^{(i)})_{i \in \mathbb{N}}$).

Since $\theta(\alpha^{n_0}) \in O_1$ (resp. $\theta(\alpha^{(n_1)}) \in O_1$) the map ϑ necessarily sends $\theta(\alpha^{n_0})$ (resp. $\theta(\alpha^{(n_1)})$) to α^{n_0} (resp. $\alpha^{(n_1)}$).

Now if $\theta(\alpha^{n_0-1}) \in O_2$, the permutation ϑ sends $\theta(\alpha^{n_0-1})$ to α^{n_0-1} and for all $0 \leq i \leq n_0 - 1$, as long as $\theta(\alpha^i) \in O_2$ we necessarily have $\vartheta(\theta(\alpha^i)) = \alpha^i$.

Suppose that there exists $i \in \mathbb{N}$, $0 \leq i \leq n_0 - 1$, such that $\theta(\alpha^i) \in O_3$; then it satisfies the stability condition. Let i_0 be the largest such integer. Then the image of $\theta(\alpha^{i_0})$ via the map ϑ could be either α^{i_0} or $\theta(\alpha^{i_0})'$ since α^{i_0+1} has already a preimage. Since $\theta(\theta(\alpha^{i_0})') \in O_1$, the map ϑ sends $\theta(\theta(\alpha^{i_0})')$ to $\theta(\alpha^{i_0})'$. Then the permutation ϑ sends $\theta(\alpha^{i_0})$ to α^{i_0} . By decreasing induction on i , we then deduce that, for all $0 \leq i \leq n_0$, $\vartheta(\theta(\alpha^i)) = \alpha^i$. Similarly we obtain that, for all $0 \leq i \leq n_1$, $\vartheta(\theta(\alpha^{(i)})) = \alpha^{(i)}$.

It follows that if $\alpha^i \in O_3$ then $(\alpha^i)'$ is sent to $\theta((\alpha^i)')$ since α^i has already a preimage. Of course, since $\theta((\alpha^i)') \in O_1$, the latter is sent to $(\alpha^i)'$. An increasing induction on i proves then that $\vartheta(\alpha^i) = \theta(\alpha^i)$ for all $0 \leq i \leq n_0$, and it follows that, if $\theta(\alpha^i) \in O_3$, then $\theta(\alpha^i)'$ is sent to its image by θ .

Similarly one has that if $\alpha^{(i)} \in O_3$, then $(\alpha^{(i)})'$ is sent to its image by θ , $\vartheta(\alpha^{(i)}) = \theta(\alpha^{(i)})$ for all $0 \leq i \leq n_1$ and if $\theta(\alpha^{(i)}) \in O_3$ then $\theta(\alpha^{(i)})'$ is sent to its image by θ . \square

3.7. Cyclic roots. Let $\alpha \in O$. We say that α is “cyclic” and write $\alpha \in O_{cyc}$ if there exist $\beta, \gamma \in O$ such that the following conditions are satisfied :

- (i) $\theta(\alpha) + \gamma = \beta + \theta(\beta)$
- (ii) $\theta(\gamma) + \beta = \alpha + \theta(\alpha)$
- (iii) $\theta(\beta) + \alpha = \gamma + \theta(\gamma)$
- (iv) $\{\alpha, \beta, \gamma, \theta(\alpha), \theta(\beta), \theta(\gamma)\} \subset O_2 \sqcup O_3$
- (v) $|\{\alpha, \beta, \gamma, \theta(\alpha), \theta(\beta), \theta(\gamma)\}| = 6$
- (vi) If $\delta \in \{\alpha, \beta, \gamma, \theta(\alpha), \theta(\beta), \theta(\gamma)\} \cap O_3$, then there exists $\tilde{\delta} \in S_\delta$ such that $\tilde{\delta}$ satisfies the stationary condition.

For $\alpha \in O_{cyc}$, set $C_\alpha = \{\alpha, \beta, \gamma, \theta(\alpha), \theta(\beta), \theta(\gamma)\}$. Note that for $\delta \in C_\alpha \cap O_3$ then, with the above notations, $\tilde{\delta}$ is unique and $S_\delta \setminus S_\delta \cap C_\alpha = \{\tilde{\delta}\}$.

Remarks. (1) If $\alpha \in O_{cyc}$ then all roots in C_α are cyclic roots.

(2) Let $\alpha \in O_{cyc}$. If $\delta \in C_\alpha \cap O_3$, then δ may satisfy the stability condition (5.3.1, first case, for instance) but we will see (6.3, paragraph (c) for instance) that it is not always the case.

(3) If $\alpha \in O_{cyc}$ then it cannot satisfy the stationary condition nor it can be stationary, even if all roots in $C_\alpha \cap O_3$ satisfy the stability condition; in the latter case the cyclic relations (i), (ii) and (iii) imply that $\alpha^1 = \gamma$, $\alpha^2 = \beta$ and $\alpha^3 = \alpha$, hence the sequence $(\alpha^i)_{i \in \mathbb{N}}$ is not stationary.

(4) Let $\tilde{\delta} \in S_\delta$ with $\delta \in O_{cyc} \cap O_3$ such that $\tilde{\delta}$ satisfies the stationary condition. Assume that, for $i, s \in \mathbb{N}$, $0 \leq s \leq i-1$, $\theta(\tilde{\delta}^s), \tilde{\delta}^{s+1} \in O_2$. Then $\tilde{\delta}^i \notin O_{st}$. Indeed otherwise $(\tilde{\delta}^i)^{(i+1)} = \theta(\tilde{\delta}^i)^{i+1}$ would satisfy the stationary condition by remark (3) in 3.5. Then by remark (1) of 3.6 one has that $(\tilde{\delta}^i)^{(i+1)} = \tilde{\delta}^{(1)}$. But the definition of $\tilde{\delta}$ gives that $\tilde{\delta}^{(1)} = \delta$ which does not satisfy the stationary condition by (3) above. Moreover by (3) above and remark (3) of 3.5, $\tilde{\delta}^i \notin O_{cyc}$. Finally if $\tilde{\delta}^i \in O_3$, then $(\tilde{\delta}^i)' \in O_{st}$ by remark (6) in 3.6.

(5) In (4) of lemma 3.8 below, the three conditions required for a root $\alpha \in O$ are disjoint in general (by (3) and (4) above).

(6) Only two of conditions (i), (ii), (iii) above are necessary. Indeed any two of them imply the third one.

Lemma. *Let $\vartheta : O \rightarrow O$ be a permutation such that for all $\gamma \in O$, $\gamma + \vartheta(\gamma) \in S$. For $\delta \in O_{cyc} \cap O_3$, let $\tilde{\delta}$ be the root in S_δ which satisfies the stationary condition. Then ϑ exchanges $\tilde{\delta}^i$ and $\theta(\tilde{\delta}^i)$ for all $i \in \mathbb{N}$.*

Proof. This follows exactly like lemma 3.6. □

3.8. A lemma of non-degeneracy. Retain the notations and hypotheses of the beginning of Section 3.

Let S^+ (resp. S^-) be the subset of S containing those $\gamma \in S$ for which $\Gamma_\gamma \subset \Delta^+$ (resp. $\Gamma_\gamma \subset \Delta_{\pi'}^-$).

Let S^m be the subset of S containing those $\gamma \in S$, for which the Heisenberg set Γ_γ contains **both** positive and negative roots in $\Delta^+ \sqcup \Delta_{\pi'}^-$.

We have $S = S^+ \sqcup S^- \sqcup S^m$ and we set $\Gamma^\pm = \bigsqcup_{\gamma \in S^\pm} \Gamma_\gamma$, $\Gamma^m = \bigsqcup_{\gamma \in S^m} \Gamma_\gamma$; then $\Gamma = \Gamma^+ \sqcup \Gamma^- \sqcup \Gamma^m$.

For all $\gamma \in S$, recall that $\Gamma_\gamma^0 = \Gamma_\gamma \setminus \{\gamma\}$, and set $O^\pm = \bigsqcup_{\gamma \in S^\pm} \Gamma_\gamma^0$, $O^m = \bigsqcup_{\gamma \in S^m} \Gamma_\gamma^0$; we have $O = O^+ \sqcup O^- \sqcup O^m$.

Set also $\mathfrak{o}^\pm = \mathfrak{g}_{-O^\pm}$ and $\mathfrak{o}^m = \mathfrak{g}_{-O^m}$ so that $\mathfrak{o} = \mathfrak{g}_{-O} = \mathfrak{o}^+ \oplus \mathfrak{o}^- \oplus \mathfrak{o}^m$.

Lemma. *Retain the above notations and hypotheses. Assume further that :*

- (1) $S|_{\mathfrak{h}_\Lambda}$ is a basis for \mathfrak{h}_Λ^* .
- (2) Let $\alpha \in \Gamma_\gamma^0$, with $\gamma \in S^+$. If there exists $\beta \in S_\alpha \cap O^+$, then $\beta \in \Gamma_\gamma^0$ and $\alpha + \beta = \gamma$ (that is, $\beta = \theta(\alpha)$).
- (3) Let $\alpha \in \Gamma_\gamma^0$, with $\gamma \in S^-$. If there exists $\beta \in S_\alpha \cap O^-$, then $\beta \in \Gamma_\gamma^0$ and $\alpha + \beta = \gamma$ (that is, $\beta = \theta(\alpha)$).
- (4) Let $\alpha \in O$ such that $S_\alpha \cap O^m \neq \emptyset$. Then either $\alpha \in O_{st}$ or $\alpha \in O_{cyc}$ or there exists $i \in \mathbb{N}$ and $\beta \in O_{cyc} \cap O_3$ such that $\tilde{\beta}^i = \theta(\alpha)$ or $\tilde{\beta}^i = \alpha$, where $\tilde{\beta}$ is the unique root in S_β that satisfies the stationary condition.

Then the restriction of the bilinear form Φ_y to $\mathfrak{o} \times \mathfrak{o}$ is non-degenerate.

Proof. Let ρ be the linear form on \mathfrak{h}^* defined by $\rho(\alpha) = 1$ for all $\alpha \in \pi$ and define $z(t) = \sum_{\gamma \in S} t^{|\rho(\gamma)|} a_\gamma x_\gamma$ for $t \in k$.

Set $d(t) = \det(\Phi_{z(t)|_{\mathfrak{o} \times \mathfrak{o}}})$ which is a polynomial in t and let H_Λ denote the adjoint group of \mathfrak{h}_Λ .

Since by hypothesis (1) the elements of $S|_{\mathfrak{h}_\Lambda}$ are linearly independent it follows that $z(ct_0)$ and $z(t_0)$ are in the same H_Λ -coadjoint orbit for $t_0 \in k$ and for all $c \in k \setminus \{0\}$. Moreover $\mathfrak{o} \times \mathfrak{o}$ is stable under the adjoint action of H_Λ . Then the degeneracy of the restriction of the bilinear form $\Phi_{z(t_0)}$ on $\mathfrak{o} \times \mathfrak{o}$ is equivalent to the degeneracy of the restriction of the bilinear form $\Phi_{z(ct_0)}$ on $\mathfrak{o} \times \mathfrak{o}$ for all $c \in k \setminus \{0\}$, that is, $d(t_0) = 0$ is equivalent to $d(ct_0) = 0$ for all $c \in k \setminus \{0\}$. It follows that either $d(t)$ is identically zero or it annihilates only at $t = 0$. Hence $d(t)$ is a multiple of a single power of t (see also [15, Rem. 8.4]).

Let $\alpha \in O$ be such that $S_\alpha \cap O^m \neq \emptyset$.

Assume first that $\alpha \in O_{st}$. Then by lemma 3.6, the only factor involving α and $\theta(\alpha)$ in $\det(\Phi_{z(t)|_{\mathfrak{o} \times \mathfrak{o}}})$ is $t^{2|\rho(\alpha+\theta(\alpha))|}$.

Secondly assume that $\alpha \in O_{cyc}$. Consider $C_\alpha = \{\alpha, \beta, \gamma, \theta(\alpha), \theta(\beta), \theta(\gamma)\}$ verifying conditions (i)-(vi) of Section 3.7. Then the matrix of $\Phi_{z(t)|_{\mathfrak{g}-C_\alpha \times \mathfrak{g}-C_\alpha}}$ is, up to a nonzero scalar, of the form

$$\begin{pmatrix} 0 & 0 & 0 & t^{|\rho(s_1)|} & t^{|\rho(s_3)|} & 0 \\ 0 & 0 & 0 & 0 & t^{|\rho(s_2)|} & t^{|\rho(s_1)|} \\ 0 & 0 & 0 & t^{|\rho(s_2)|} & 0 & t^{|\rho(s_3)|} \\ -t^{|\rho(s_1)|} & 0 & -t^{|\rho(s_2)|} & 0 & 0 & 0 \\ -t^{|\rho(s_3)|} & -t^{|\rho(s_2)|} & 0 & 0 & 0 & 0 \\ 0 & -t^{|\rho(s_1)|} & -t^{|\rho(s_3)|} & 0 & 0 & 0 \end{pmatrix}$$

where $s_1 = \alpha + \theta(\alpha)$, $s_2 = \beta + \theta(\beta)$ and $s_3 = \gamma + \theta(\gamma)$.

Hence up to a nonzero scalar,

$$\det(\Phi_{z(t)|_{\mathfrak{g}-C_\alpha \times \mathfrak{g}-C_\alpha}}) = t^{2(|\rho(s_1)|+|\rho(s_2)|+|\rho(s_3)|)} = t^{2(|\rho(\alpha+\theta(\alpha))|+|\rho(\beta+\theta(\beta))|+|\rho(\gamma+\theta(\gamma))|)}$$

and by lemma 3.7, it follows that the only factor involving α and $\theta(\alpha)$ in $\det(\Phi_{z(t)|_{\mathfrak{o} \times \mathfrak{o}}})$ is $t^{2|\rho(\alpha+\theta(\alpha))|}$.

Finally if there exists $\beta \in O_{cyc} \cap O_3$ such that $\alpha = \tilde{\beta}^i$ or $\theta(\alpha) = \tilde{\beta}^i$ for $i \in \mathbb{N}$ and $\tilde{\beta} \in S_\beta$ satisfying the stationary condition then, by lemma 3.7, in the expansion of the determinant of $\Phi_{z(t)|_{\mathfrak{o} \times \mathfrak{o}}}$ the only factor involving α and $\theta(\alpha)$ is $t^{2|\rho(\alpha+\theta(\alpha))|}$.

Let now $\alpha \in O^\pm$ such that $S_\alpha \cap O^\mp \neq \emptyset$ and $\beta \in S_\alpha \cap O^\mp$. By the above, if there is a factor in $\det(\Phi_{z(t)|_{\mathfrak{o} \times \mathfrak{o}}})$ which involves α and β , that is, if $t^{|\rho(\alpha+\beta)|}$ appears as a factor in $\det(\Phi_{z(t)|_{\mathfrak{o} \times \mathfrak{o}}})$, then necessarily $S_\alpha \cap O^m = \emptyset$ and $S_{\theta(\alpha)} \cap O^m = \emptyset$.

Then observe that $|\rho(\alpha + \beta)| < |\rho(\alpha)| + |\rho(\beta)|$,

whilst $|\rho(\alpha + \theta(\alpha))| = |\rho(\alpha)| + |\rho(\theta(\alpha))|$ and $|\rho(\beta + \theta(\beta))| = |\rho(\beta)| + |\rho(\theta(\beta))|$.

Since $d(t)$ is a multiple of a single power of t , the above observations and conditions (2) and (3) imply that $t^{|\rho(\alpha+\beta)|}$ cannot appear as a factor in $\det(\Phi_{z(t)|_{\mathfrak{o} \times \mathfrak{o}}})$. (See also the proof of [15, Lemma 8.5]).

Denote by \tilde{O} a choice of representatives in O modulo the involution θ . Then, up to a nonzero scalar,

$$d(t) = \det(\Phi_{z(t)|_{\mathfrak{o} \times \mathfrak{o}}}) = \prod_{\alpha \in \tilde{O}} t^{2|\rho(\alpha+\theta(\alpha))|}$$

Thus $\det(\Phi_{z(t)|_{\mathfrak{o} \times \mathfrak{o}}}) \neq 0$ for $t \neq 0$ and the assertion of the lemma follows. \square

Remark. If $S^m = \emptyset$ then condition (4) is empty and the above lemma is [8, Lemma 5].

3.9. By Lemmata 3.1 and 3.8 we obtain the following corollary :

Corollary. *Assume that the hypotheses of the previous lemma hold and that $|T| = \text{ind } \mathfrak{p}_{\pi', \Lambda}$, where $T = (\Delta^+ \sqcup \Delta_{\pi'}^-) \setminus \Gamma$. Recall that $y = \sum_{\gamma \in S} a_{\gamma} x_{\gamma}$ and define $h \in \mathfrak{h}_{\Lambda}$ by $\gamma(h) = -1$ for all $\gamma \in S$. Then (h, y) is an adapted pair for $\mathfrak{p}_{\pi', \Lambda}^-$.*

In what follows, we construct adapted pairs for the truncated maximal parabolic subalgebras \mathfrak{p} in type B, D, E_6 where the lower and upper bounds of Section 2.7 do not coincide; in type B and D, \mathfrak{p} is associated to the subsystem π' of π obtained by suppressing a root of even index. The construction of an adapted pair in these cases is much more involved than in [8].

4. TYPE B

In this section, \mathfrak{g} is a simple Lie algebra of type B_n ($n \geq 2$) and $\mathfrak{p} = \mathfrak{p}_{\pi', \Lambda}^-$ is the truncated maximal parabolic subalgebra associated to the subset $\pi' = \pi \setminus \{\alpha_s\}$ of π with s even ($2 \leq s \leq n$). In this case the lower and the upper bounds in 2.7 for $\text{ch } Y(\mathfrak{p})$ do not coincide [8, 4.1] (except when $n = s = 2$ or $n = s = 4$, cases that we will however also consider in the following).

We will construct an adapted pair (h, y) for \mathfrak{p} , a slice for its coadjoint action and show that $Y(\mathfrak{p})$ is polynomial in $\text{ind } \mathfrak{p}$ generators. It will follow by the discussion in the introduction that the field $C(\mathfrak{p}_{\pi'}^-)$ of invariant fractions is a purely transcendental extension of k .

As we said above, it is enough to find sets S, T that satisfy the conditions of lemma 3.8 and of corollary 3.9.

Recall that the truncated Cartan subalgebra of \mathfrak{p} is the Cartan subalgebra of the Levi factor, namely $\mathfrak{h}_{\Lambda} = \mathfrak{h}' = \bigoplus_{1 \leq i \leq n, i \neq s} k \alpha_i^{\vee}$.

4.1. The set S . Denote by $\{\varepsilon_i \mid 1 \leq i \leq n\}$ an orthonormal basis of \mathbb{R}^n according to which the simple roots α_i ($1 \leq i \leq n$) of \mathfrak{g} are expanded as in [1, Planche II]. Then the strongly orthogonal positive roots given in [8, Table I] are $\beta_i = \varepsilon_{2i-1} + \varepsilon_{2i}$ for all $1 \leq i \leq [n/2]$, and for n odd, $\beta_{(n+1)/2} = \varepsilon_n = \alpha_n$ and $\beta_{i'} = \alpha_{2i'-1} = \varepsilon_{2i'-1} - \varepsilon_{2i'}$ for $1 \leq i' \leq [n/2]$.

If $n = s$, set $S^+ = \{\beta_i \mid 1 \leq i \leq s/2 - 1\}$

and if $n > s$, set $S^+ = \{\beta_i, \varepsilon_{s-1} + \varepsilon_{s+1}, \varepsilon_{2j} + \varepsilon_{2j+1} \mid 1 \leq i \leq s/2 - 1, s/2 + 1 \leq j \leq [(n-1)/2]\}$.

Set $S^- = \{\varepsilon_{s-i} - \varepsilon_i, -\beta_j = -\varepsilon_{2j-1} - \varepsilon_{2j} \mid 1 \leq i \leq s/2 - 1, s/2 + 1 \leq j \leq [n/2]\}$.

Finally set $S^m = \{\varepsilon_s\}$ and $S = S^+ \sqcup S^- \sqcup S^m$.

Clearly, $S \subset \Delta^+ \sqcup \Delta_{\pi}^-$ and $|S| = n - 1 = \dim \mathfrak{h}_{\Lambda}$. We first show below that condition (1) of lemma 3.8 holds.

Lemma. $S|_{\mathfrak{h}_{\Lambda}}$ is a basis for \mathfrak{h}_{Λ}^* .

Proof. It is sufficient to show that if $S = \{s_1, \dots, s_{n-1}\}$ and $\{h_1, \dots, h_{n-1}\}$ is a basis of \mathfrak{h}_{Λ} , then $\det(s_i(h_j))_{i,j} \neq 0$. We will prove this statement by induction on n . We choose $\{\alpha_i^{\vee} \mid 1 \leq i \leq n, i \neq s\}$ as a basis of \mathfrak{h}_{Λ} .

Add temporarily a lower subscript n to S^{\pm} , π , $\mathfrak{h}' = \mathfrak{h}_{\Lambda}$ to emphasize that they are defined for type B_n and observe that S^m does not depend on n .

Identify an element $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ with the element $(x_1, x_2, \dots, x_n, 0) \in \mathbb{R}^{n+1}$. Observe that $S_{s+1}^+ = S_s^+ \sqcup \{\varepsilon_{s-1} + \varepsilon_{s+1}\}$, whereas for n even and $n \geq s + 2$ we have $S_{n+1}^+ = S_n^+ \sqcup \{\varepsilon_n + \varepsilon_{n+1}\}$ and for n odd we have $S_{n+1}^+ = S_n^+$.

Similarly for n even, $S_{n+1}^- = S_n^-$ and for n odd, $S_{n+1}^- = S_n^- \sqcup \{-\varepsilon_n - \varepsilon_{n+1}\}$. Finally set $S_n = S_n^+ \sqcup S_n^- \sqcup S^m$.

We first consider the case $n = s$.

If $n = s = 2$ then $S = \{s_1 = \varepsilon_2 = \alpha_2\}$ and $\det(s_1(\alpha_1^{\vee})) = -1 \neq 0$.

Assume now that $n \geq 4$ and $n = s$. Then $S = \{\varepsilon_n, \varepsilon_{n-i} - \varepsilon_i, \beta_j \mid 1 \leq i, j \leq n/2 - 1\}$. Recall that $\{\varpi_i\}_{1 \leq i \leq n}$ is the set of fundamental weights of \mathfrak{g} . One has that for all i , with $1 \leq i \leq n/2 - 1$, $\beta_i = \varpi_{2i} - \varpi_{2i-2}$ (where we have set $\varpi_0 = 0$) and $\varepsilon_n = -\varpi_{n-1} + 2\varpi_n$. Also, for all i , with $1 \leq i \leq n/2 - 1$, $\varepsilon_{n-i} - \varepsilon_i = -\varpi_i + \varpi_{i-1} - \varpi_{n-1-i} + \varpi_{n-i}$.

Then, by ordering the basis of \mathfrak{h}_{Λ} as

$\{\alpha_2^{\vee}, \alpha_4^{\vee}, \dots, \alpha_{n-2}^{\vee}, \alpha_{n-1}^{\vee}, \alpha_1^{\vee}, \alpha_{n-3}^{\vee}, \alpha_3^{\vee}, \dots, \alpha_{n/2+1}^{\vee}, \alpha_{n/2-1}^{\vee}\}$ if $n/2$ is even,

and as $\{\alpha_2^{\vee}, \alpha_4^{\vee}, \dots, \alpha_{n-2}^{\vee}, \alpha_{n-1}^{\vee}, \alpha_1^{\vee}, \alpha_{n-3}^{\vee}, \alpha_3^{\vee}, \dots, \alpha_{n/2-2}^{\vee}, \alpha_{n/2}^{\vee}\}$ if $n/2$ is odd, and

by ordering elements of S as $\{\beta_1, \beta_2, \dots, \beta_{n/2-1}, \varepsilon_n, \varepsilon_{n-1} - \varepsilon_1, \varepsilon_{n-2} - \varepsilon_2, \dots, \varepsilon_{n/2+1} - \varepsilon_{n/2-1}\}$, we have that $(s_i(h_j))_{ij} = \begin{pmatrix} A & 0 \\ C & D \end{pmatrix}$ where A is an $(n/2) \times (n/2)$ lower triangular matrix with 1 everywhere on the diagonal, except the last element which is equal to -1 and D is a $(n/2 - 1) \times (n/2 - 1)$ lower triangular matrix with -1 everywhere on the diagonal. Hence $\det(s_i(h_j))_{ij} = (-1)^{n/2} \neq 0$.

For every $n \geq s$, let $\{h_1, \dots, h_{n-1}, h_n\}$ be a basis for the truncated Cartan \mathfrak{h}'_{n+1} of the truncated parabolic associated to $\pi_{n+1} \setminus \{\alpha_s\}$ in type B_{n+1} , such that $\{h_1, \dots, h_{n-1}\}$ is a basis of the truncated Cartan \mathfrak{h}'_n for the truncated parabolic associated to $\pi_n \setminus \{\alpha_s\}$ in type B_n with the identification in the beginning of this proof.

Then, using the observation in the beginning of this proof, and ordering the elements of $S_{n+1} = \{s_1, s_2, \dots, s_n\}$ such that its first $n-1$ elements are those of S_n , we get that $\det(s_i(h_j))_{1 \leq i, j \leq n} = (-1)^n \det(s_i(h_j))_{1 \leq i, j \leq n-1}$, which completes the proof of the lemma. \square

4.2. The Heisenberg sets Γ_γ for all $\gamma \in S$. Recall the (maximal in Δ^+) Heisenberg set H_{β_i} of centre β_i defined in [8, Lemma 3] for every strongly orthogonal positive root β_i .

4.2.1. The set Γ^m . We set

$$\Gamma_{\varepsilon_s} = \{\varepsilon_s, \varepsilon_i, \varepsilon_s - \varepsilon_i, \varepsilon_s + \varepsilon_j, -\varepsilon_j \mid 1 \leq i \leq n, i \neq s, s+1 \leq j \leq n\} \subset \Delta^+ \sqcup \Delta_{\pi'}^-.$$

4.2.2. The set Γ^+ . For all i , with $1 \leq i \leq s/2-1$ we set $\Gamma_{\beta_i} = H_{\beta_i} \setminus \{\varepsilon_{2i-1}, \varepsilon_{2i}\} \subset \Delta^+$.

Set $\Gamma_{\varepsilon_{s-1}+\varepsilon_{s+1}} = \{\varepsilon_{s-1} + \varepsilon_{s+1}, \varepsilon_{s-1} \pm \varepsilon_i, \varepsilon_{s+1} \mp \varepsilon_i \mid s+2 \leq i \leq n\} \subset \Delta^+$.

For all i , with $s/2+1 \leq i \leq [(n-1)/2]$, set $\Gamma_{\varepsilon_{2i}+\varepsilon_{2i+1}} = \{\varepsilon_{2i} + \varepsilon_{2i+1}, \varepsilon_{2i} \pm \varepsilon_j, \varepsilon_{2i+1} \mp \varepsilon_j \mid 2i+2 \leq j \leq n\} \subset \Delta^+$.

4.2.3. The set Γ^- . For all i , with $1 \leq i \leq s/2-1$, set $\Gamma_{\varepsilon_{s-i}-\varepsilon_i} = \{\varepsilon_{s-i} - \varepsilon_i, \varepsilon_j - \varepsilon_i, \varepsilon_{s-i} - \varepsilon_j \mid i+1 \leq j \leq s-i-1\} \subset \Delta_{\pi'}^-$.

For all i , with $s/2+1 \leq i \leq [n/2]$, set $\Gamma_{-\varepsilon_{2i-1}-\varepsilon_{2i}} = \{-\varepsilon_{2i-1} - \varepsilon_{2i}, -\varepsilon_{2i-1} \pm \varepsilon_j, -\varepsilon_{2i} \mp \varepsilon_j \mid 2i+1 \leq j \leq n\} \subset \Delta_{\pi'}^-$.

By construction, the sets Γ_γ , $\gamma \in S$, are disjoint Heisenberg sets of centre γ , included in $\Delta^+ \sqcup \Delta_{\pi'}^-$.

4.3. Conditions (2), (3) and (4) of lemma 3.8. First, one may check that $\alpha \in O \setminus O_1 \sqcup O_2$ only if $\alpha = \varepsilon_i - \varepsilon_j \in O^-$ with $1 \leq j < s/2 < i \leq s-1-j$. But in this case $S_{\varepsilon_i-\varepsilon_j} \cap O^- = \{\theta(\varepsilon_i - \varepsilon_j)\}$ and $S_{\varepsilon_i-\varepsilon_j} \cap O^m = \emptyset$. Hence conditions (3) and (4) are satisfied for such a root. Moreover for $i, j \neq s+1$, $\varepsilon_i + \varepsilon_j \in O_1$ unless $\varepsilon_i + \varepsilon_j = \varepsilon_{s-1} + \varepsilon_s \in T$ (where T is the complement of $\Gamma = \sqcup_{\gamma \in S} \Gamma_\gamma$ in $\Delta^+ \sqcup \Delta_{\pi'}^-$) and $\varepsilon_s + \varepsilon_{s+1}, \varepsilon_s - \varepsilon_{s-1}, \varepsilon_s - \varepsilon_{s+1} \in O_1$.

Denote by π'_1 the connected component of π' of type A_{s-1} and π'_2 the connected component of π' of type B_{n-s} . Observe that for all i , with $1 \leq i \leq s/2-1$, $\Gamma_{\varepsilon_{s-i}-\varepsilon_i} \subset \Delta_{\pi'_1}^-$ and for all i , with $s/2+1 \leq i \leq [n/2]$, $\Gamma_{-\varepsilon_{2i-1}-\varepsilon_{2i}} \subset \Delta_{\pi'_2}^-$.

4.3.1. Let $\alpha \in O^\pm$ and assume that there exists $\beta \in O^\pm \sqcup O^m$ such that $\alpha + \beta \in S$. Below we verify conditions (2) and (4) if $\alpha \in O^+$ and conditions (3) and (4) if $\alpha \in O^-$.

First case: $\alpha \in \Gamma_{\beta_i}^0 \subset O^+$, with $1 \leq i \leq s/2 - 1$.

Assume first that $\beta \in \Gamma_{\beta_j}^0 \subset \Delta^+$, with $1 \leq j \leq s/2 - 1$. Then by [8, Lemma 3 (5)] $\alpha + \beta \in H_{\beta_k}$ where $k = \min\{i, j\}$. But the only element of S in H_{β_k} is β_k . By [8, Lemma 3 (5)] again, it follows that $\alpha, \beta \in H_{\beta_k} \setminus \{\beta_k\}$. Hence $i = j = k$.

Assume now that $\beta \in O^+ \sqcup O^m$ but $\beta \notin \bigsqcup_{j=1}^{s/2-1} \Gamma_{\beta_j}^0$. If $\beta \in \Delta^+$, either $\beta \in \{\varepsilon_j \mid 1 \leq j \leq s-2\}$ or by [8, Lemma 3 (2)], $\beta \in \bigsqcup_{j=s/2}^{[(n+1)/2]} H_{\beta_j} \sqcup \bigsqcup_{j'=s/2+1}^{[n/2]} H_{\beta_{j'}}$, where recall that $H_{\beta_{j'}} = \{\beta_{j'}\} = \{\alpha_{2j'-1}\}$. By a similar reasoning as before or an easy computation one shows that this is not possible. Hence condition (2) for α is satisfied.

Finally suppose that $\beta \in O^m \cap \Delta^-$ and then $S_\alpha \cap O^m \neq \emptyset$. One verifies that $\alpha = \varepsilon_j - \varepsilon_s$ and $\beta = \varepsilon_s - \varepsilon_{s-j}$, with $j \in \{2i-1, 2i\}$ and $j > s/2$. Moreover one verifies that $\theta(\alpha) = \varepsilon_{j+1} + \varepsilon_s \in O_1$ if j is odd and $\theta(\alpha) = \varepsilon_{j-1} + \varepsilon_s \in O_1$ if j is even. We will assume that j is odd; the other case is very similar.

Recall the sequences of roots in O constructed from a root in O (3.3). Since $\theta(\alpha) \in O_1$, we have that the sequence $(\alpha^k)_{k \in \mathbb{N}}$ constructed from α is stationary at rank 0. We will determine the sequence $(\alpha^{(k)})_{k \in \mathbb{N}}$ constructed from $\theta(\alpha)$. Recall that $\alpha^{(0)} = \theta(\alpha)$, then since $\alpha = \theta(\alpha^{(0)}) \in O_2$, we necessarily have $\alpha^{(1)} = \beta = \varepsilon_s - \varepsilon_{s-j}$. One has that $\theta(\beta) = \varepsilon_{s-j} \in O_2$, with $S_{\varepsilon_{s-j}} = \{\beta, \varepsilon_{s-j+1}\}$. Then $\alpha^{(2)} = \varepsilon_{s-j+1} \in O^m$ and $\theta(\alpha^{(2)}) = \varepsilon_s - \varepsilon_{s-j+1}$. Then $\alpha^{(3)} = \varepsilon_{j-1} - \varepsilon_s$ and $\theta(\alpha^{(3)}) = \varepsilon_{j-2} + \varepsilon_s \in O_1$ (unless $j = s/2 + 1$ in which case already $\theta(\alpha^{(2)}) \in O_1$). We conclude that the sequence $(\alpha^{(k)})_{k \in \mathbb{N}}$ is stationary at rank at most 3. On the other hand, all the roots involved here belong to $O_1 \sqcup O_2$. Hence $\alpha \in O_{st}$ since the roots α and $\theta(\alpha)$ satisfy the stationary condition. Hence condition (4) is satisfied for such an α .

Second Case: $\alpha \in \Gamma_{\varepsilon_{s-1} + \varepsilon_{s+1}}^0 \subset O^+$.

If $\beta \in \Gamma_{\varepsilon_{s-1} + \varepsilon_{s+1}}^0$ then necessarily $\alpha + \beta = \varepsilon_{s-1} + \varepsilon_{s+1}$ thus $\beta = \theta(\alpha)$.

For condition (2), it remains to check the case where $\beta \in \bigsqcup_{j=s/2+1}^{[(n-1)/2]} \Gamma_{\varepsilon_{2j} + \varepsilon_{2j+1}}^0$. But then it is not possible that $\alpha + \beta \in S$ since α contains ε_{s-1} or ε_{s+1} and β contains ε_i with $i \geq s+2$, while in S^+ there is no root containing a linear combination of both ε_{s-1} or ε_{s+1} and ε_i with $i \geq s+2$.

Finally, for condition (4) one easily checks that it is not possible that $\beta \in \Gamma_{\varepsilon_s}^0$.

Third case: $\alpha \in \Gamma_{\varepsilon_{2i} + \varepsilon_{2i+1}}^0 \subset O^+$, with $s/2 + 1 \leq i \leq [(n-1)/2]$.

For condition (2), it remains to check the case where $\beta \in \Gamma_{\varepsilon_{2j} + \varepsilon_{2j+1}}^0$, with $s/2 + 1 \leq j \leq [(n-1)/2]$. Then one has that $i = j$ and $\alpha + \beta = \varepsilon_{2i} + \varepsilon_{2i+1}$, thus $\beta = \theta(\alpha)$.

Finally, for condition (4) one checks that it is not possible that $\beta \in \Gamma_{\varepsilon_s}^0$.

Fourth case: $\alpha \in \Gamma_{\varepsilon_{s-i}-\varepsilon_i}^0 \subset O^- \cap \Delta_{\pi_1}^-$, with $1 \leq i \leq s/2 - 1$.

Assume first that $\beta \in \Gamma_{\varepsilon_{s-j}-\varepsilon_j}^0$, with $1 \leq j \leq s/2 - 1$. Since $\Gamma_{\varepsilon_{s-i}-\varepsilon_i}^0 \subset \Delta_{\pi_1}^-$ and $\Gamma_{\varepsilon_{s-j}-\varepsilon_j}^0 \subset \Delta_{\pi_1}^-$, one has that $\alpha + \beta \in \Delta_{\pi_1}^-$ and so there exists k , with $1 \leq k \leq s/2 - 1$ such that $\alpha + \beta = \varepsilon_{s-k} - \varepsilon_k$. Then necessarily $i = j = k$, thus $\beta = \theta(\alpha)$.

On the other hand, it is not possible that $\beta \in \Gamma_{-\varepsilon_{2j-1}-\varepsilon_{2j}}^0$, with $s/2 + 1 \leq j \leq [n/2]$, since in that case $\beta \in \Delta_{\pi_2}^-$ and so $\alpha + \beta \notin \Delta$. Hence condition (3) for α is satisfied.

Finally one also checks that it is not possible that $\beta \in \Gamma_{\varepsilon_s}^0$.

Fifth case: $\alpha \in \Gamma_{-\varepsilon_{2i-1}-\varepsilon_{2i}}^0 \subset O^- \cap \Delta_{\pi_2}^-$, with $s/2 + 1 \leq i \leq [n/2]$.

For condition (3), it remains to check the case where $\beta \in \Gamma_{-\varepsilon_{2j-1}-\varepsilon_{2j}}^0$, with $s/2 + 1 \leq j \leq [n/2]$ then there exists k , with $s/2 + 1 \leq k \leq [n/2]$ such that $\alpha + \beta = -\varepsilon_{2k-1} - \varepsilon_{2k}$ (since $\alpha + \beta \in \Delta_{\pi_2}^- \cap S$) and one checks that $i = j = k$, thus $\beta = \theta(\alpha)$.

Finally one checks that it is not possible that $\beta \in \Gamma_{\varepsilon_s}^0$.

4.3.2. Assume that $\alpha \in O^m (= \Gamma_{\varepsilon_s}^0)$. Note that $S_\alpha \cap O^m \neq \emptyset$, since it contains $\theta(\alpha)$. We will show that $\alpha \in O_{st}$ and so condition (4) holds.

Recall that $\Gamma_{\varepsilon_s}^0 = \{\varepsilon_i, \varepsilon_s - \varepsilon_i, -\varepsilon_j, \varepsilon_s + \varepsilon_j \mid 1 \leq i \leq n, i \neq s, s+1 \leq j \leq n\}$. We will show that the sequences constructed by the roots in $\Gamma_{\varepsilon_s}^0$ are stationary and lie in $O_1 \sqcup O_2$. Note that this is enough to prove our main claim.

For $\alpha = -\varepsilon_j$, with $s+1 \leq j \leq n$, we have that $\theta(\alpha) \in O_1$, hence the sequence (α^i) is stationary at rank 0.

For $\alpha = \varepsilon_s + \varepsilon_j$, with $s+1 \leq j \leq n$, we have $\theta(\alpha) \in O_2$ and $\alpha^1 = -\varepsilon_{j+1}$ if j is odd and $\alpha^1 = -\varepsilon_{j-1}$ if j is even. Then $\theta(\alpha^1) \in O_1$, hence (α^i) is stationary at rank 1.

For $\alpha = \varepsilon_i$ with $i \geq s+2$ then $\theta(\alpha) \in O_1$. Also for $\alpha = \varepsilon_s - \varepsilon_i$ and $i \geq s+2$, $\theta(\alpha) \in O_2$ and $\alpha^1 = \varepsilon_{i+1}$ if i even, $\alpha^1 = \varepsilon_{i-1}$ if i odd and $\theta(\alpha^1) \in O_1$. We conclude as above.

For $\alpha = \varepsilon_{s\pm 1}$, then $\theta(\alpha) \in O_1$ and we are done. For $\alpha = \varepsilon_s - \varepsilon_{s\pm 1}$ then $\theta(\alpha) = \varepsilon_{s\pm 1} \in O_2$ and $\alpha^1 = \varepsilon_{s\mp 1}$ is such that $\theta(\alpha^1) = \varepsilon_s - \varepsilon_{s\mp 1} \in O_1$.

Finally, for $\alpha = \varepsilon_i$ with $1 \leq i \leq s-2$, $\theta(\alpha) \in O_1$ if $i \geq s/2$ otherwise $\theta(\alpha) \in O_2$. In the latter case, the first case of 4.3.1 (last part) and remark (3) of 3.6 allow us to conclude that $\theta(\alpha) \in O_{st}$ (hence also $\alpha \in O_{st}$).

In the former case, it remains to consider $\alpha = \varepsilon_s - \varepsilon_i$, with $s/2 \leq i \leq s-2$. Then $\theta(\alpha) = \varepsilon_i \in O_2$ and $\alpha^1 = \varepsilon_{i+1}$ if i is odd, $\alpha^1 = \varepsilon_{i-1}$ if i is even and $\theta(\alpha^1) \in O_1$ (unless $i = s/2$ and is even, in which case $\theta(\alpha^1) \in O_2$ and by the above, $\theta(\alpha^1) \in O_{st}$ and then $\alpha \in O_{st}$).

4.4. **The set T .** Recall that we denote by T the complement of $\Gamma = \Gamma^+ \sqcup \Gamma^- \sqcup \Gamma^m$ in $\Delta^+ \sqcup \Delta_{\pi}^-$.

One easily calculates that for $n = s$, $T = \{\varepsilon_{s-1} + \varepsilon_s, \varepsilon_{2i-1} - \varepsilon_{2i} \mid 1 \leq i \leq s/2\}$ and for $n > s$, $T = \{\varepsilon_{s-1} + \varepsilon_s, \varepsilon_{s-1} - \varepsilon_{s+1}, \varepsilon_{2i-1} - \varepsilon_{2i}, -\varepsilon_{s+2j-1} + \varepsilon_{s+2j}, \varepsilon_{s+2k} - \varepsilon_{s+2k+1} \mid 1 \leq i \leq s/2, 1 \leq j \leq [(n-s)/2], 1 \leq k \leq [(n-s-1)/2]\}$.

It follows that $|T| = n - s/2 + 1$. On the other hand, recall that the index of \mathfrak{p} equals the number of $\langle \mathbf{ij} \rangle$ -orbits in π where \mathbf{i} and \mathbf{j} are the involutions of π of Section 2.7.

Here the $\langle \mathbf{ij} \rangle$ -orbits in π are $\Gamma_t = \{\alpha_t, \alpha_{s-t}\}$ for $1 \leq t \leq s/2 - 1$, $\Gamma_{s/2} = \{\alpha_{s/2}\}$ and $\Gamma_t = \{\alpha_t\}$ for $s \leq t \leq n$. They are $n - s/2 + 1$ in number hence $\text{ind } \mathfrak{p} = n - s/2 + 1$.

Remark. All conditions of lemma 3.1 are satisfied. Hence by defining $h \in \mathfrak{h}_\Lambda$ by $\gamma(h) = -1$, for all $\gamma \in S$ and by setting $y = \sum_{\gamma \in S} x_\gamma$ we obtain an adapted pair (h, y) for $\mathfrak{p}_{\pi', \Lambda}^-$.

4.5. The eigenvalues of $\text{ad } h$ on \mathfrak{g}_T . The semisimple element h of the adapted pair is uniquely defined by the relations $\gamma(h) = -1$ for all $\gamma \in S$. In this paragraph, we compute the values of h on the elements of T , that is the $\text{ad } h$ eigenvalues on a complement of the $\text{ad } \mathfrak{p}_{\pi', \Lambda}^-$ -orbit of y .

By a direct computation, one verifies that

$$\begin{aligned} h &= \sum_{k=1}^{[s/4]} \left(\frac{s}{2} + 2k - 1 \right) \varepsilon_{2k-1} + \sum_{k=[s/4]+1}^{s/2-1} \left(\frac{3s}{2} - 2k \right) \varepsilon_{2k-1} \\ &\quad - \sum_{k=1}^{[s/4]} \left(\frac{s}{2} + 2k \right) \varepsilon_{2k} - \sum_{k=[s/4]+1}^{s/2-1} \left(\frac{3s}{2} + 1 - 2k \right) \varepsilon_{2k} + \frac{s}{2} \varepsilon_{s-1} - \varepsilon_s \\ &\quad + \sum_{k=1}^{[(n-s+1)/2]} \left(-2k + 1 - \frac{s}{2} \right) \varepsilon_{s+2k-1} + \sum_{k=1}^{[(n-s)/2]} \left(2k + \frac{s}{2} \right) \varepsilon_{s+2k}. \end{aligned}$$

Then the eigenvalues of $\text{ad } h$ on \mathfrak{g}_T are :

- $s + 4i - 1 = (\varepsilon_{2i-1} - \varepsilon_{2i})(h)$ for all i , with $1 \leq i \leq [s/4]$.
- $3s - 4i + 1 = (\varepsilon_{2i-1} - \varepsilon_{2i})(h)$ for all i , with $[s/4] + 1 \leq i \leq s/2 - 1$.
- $s/2 + 1 = (\varepsilon_{s-1} - \varepsilon_s)(h)$.
- $s/2 - 1 = (\varepsilon_{s-1} + \varepsilon_s)(h)$.
- $s + 1 = (\varepsilon_{s-1} - \varepsilon_{s+1})(h)$.
- $s + 4j - 1 = (-\varepsilon_{s+2j-1} + \varepsilon_{s+2j})(h)$, for all j , with $1 \leq j \leq [(n-s)/2]$.
- $s + 4j + 1 = (\varepsilon_{s+2j} - \varepsilon_{s+2j+1})(h)$, for all j , with $1 \leq j \leq [(n-s-1)/2]$.

From the last three equalities we have that $s + 2k - 1$ is an eigenvalue of $\text{ad } h$ on \mathfrak{g}_T , for all k , with $1 \leq k \leq n - s$.

4.6. Polynomiality of $Y(\mathfrak{p})$. Recall the bounds for $\text{ch } Y(\mathfrak{p})$ in Section 2.7 as well as the improved upper bound given an adapted pair in Section 2.9. We will show that the lower bound and the improved upper bound coincide, hence $Y(\mathfrak{p})$ is a polynomial algebra over k .

4.6.1. *The lower bound for $\text{ch } Y(\mathfrak{p})$.* The lower bound for $\text{ch } Y(\mathfrak{p})$ is

$$\prod_{\Gamma \in E(\pi')} (1 - e^{\delta_\Gamma})^{-1} \leq \text{ch } Y(\mathfrak{p}).$$

We will compute it explicitly. As we already said in Section 4.4, the set of $\langle \mathbf{ij} \rangle$ -orbits in π is

$$E(\pi') = \{\Gamma_{s/2} := \{\alpha_{s/2}\}, \Gamma_t := \{\alpha_t, \alpha_{s-t}\}, \Gamma_u := \{\alpha_u\} \mid 1 \leq t \leq s/2-1, s \leq u \leq n\}.$$

It remains to compute δ_Γ for each $\Gamma \in E(\pi')$.

Let $\Gamma \in E(\pi')$. Since $\mathbf{j} = \text{id}_\pi$ and $\mathbf{i}(\Gamma \cap \pi') = \mathbf{j}(\Gamma) \cap \pi'$, one has

$$\delta_\Gamma = -2\left(\sum_{\gamma \in \Gamma} \varpi_\gamma - \sum_{\gamma \in \Gamma \cap \pi'} \varpi'_\gamma\right).$$

Assume first that $n = s$. Then the Levi factor of \mathfrak{p} is of type A_{n-1} and one may check that for all $1 \leq t \leq n-1$, $\varpi_t - \varpi'_t = \frac{2t}{n} \varpi_n$. Then for all $1 \leq t \leq n/2-1$, $\delta_{\Gamma_t} = -2(\varpi_t - \varpi'_t + \varpi_{n-t} - \varpi'_{n-t}) = -4\varpi_n$ and $\delta_{\Gamma_n} = \delta_{\Gamma_{n/2}} = -2\varpi_n$. Hence, one has

$$\prod_{\Gamma \in E(\pi')} (1 - e^{\delta_\Gamma})^{-1} = (1 - e^{-2\varpi_n})^{-2} (1 - e^{-4\varpi_n})^{-(n/2-1)} \quad (1).$$

Assume now that $n > s$. Then the Levi factor of \mathfrak{p} is the product of a simple Lie algebra of type A_{s-1} and a simple Lie algebra of type B_{n-s} .

For all $1 \leq t \leq s-1$, one checks that $\varpi_t - \varpi'_t = \frac{t}{s} \varpi_s$. Then, for all $1 \leq t \leq s/2-1$, one has $\delta_{\Gamma_t} = -2\varpi_s$ and $\delta_{\Gamma_{s/2}} = -\varpi_s$. On the other hand, for all $s+1 \leq t \leq n-1$, one has that $\varpi_t - \varpi'_t = \varpi_s$, hence $\delta_{\Gamma_t} = -2\varpi_s$. Finally $\varpi_n - \varpi'_n = \frac{1}{2} \varpi_s$ and $\delta_{\Gamma_n} = -\varpi_s$, whereas $\delta_{\Gamma_s} = -2\varpi_s$, since $\Gamma_s \cap \pi' = \emptyset$.

We conclude that for $n > s$ one has

$$\prod_{\Gamma \in E(\pi')} (1 - e^{\delta_\Gamma})^{-1} = (1 - e^{-\varpi_s})^{-2} (1 - e^{-2\varpi_s})^{-(n-1-s/2)} \quad (2)$$

4.6.2. *The improved upper bound for $\text{ch } Y(\mathfrak{p})$.* Recall Section 2.9 that the improved upper bound for $\text{ch } Y(\mathfrak{p})$ is

$$\text{ch } Y(\mathfrak{p}) \leq \prod_{\gamma \in T} (1 - e^{-(\gamma + t(\gamma))})^{-1},$$

where for all $\gamma \in T$, $t(\gamma)$ is the unique element in $\mathbb{Q}S$ such that $\gamma + t(\gamma)$ is a multiple of ϖ_s . We will compute $t(\gamma)$, for all $\gamma \in T$.

Assume first that $n = s$ and recall Section 4.4 that $T = \{\varepsilon_{s-1} + \varepsilon_s, \varepsilon_{2i-1} - \varepsilon_{2i} \mid 1 \leq i \leq s/2\}$. Recall also Section 4.1 that $S = \{\varepsilon_s, \varepsilon_{s-i} - \varepsilon_i, \varepsilon_{2j-1} + \varepsilon_{2j} \mid 1 \leq i, j \leq s/2-1\}$. Finally recall that $\varpi_s = \varpi_n = 1/2(\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n)$.

By a direct calculation, one may verify that:

- $t(\varepsilon_{s-1} + \varepsilon_s) = (\varepsilon_1 + \varepsilon_2) + (\varepsilon_3 + \varepsilon_4) + \dots + (\varepsilon_{n-3} + \varepsilon_{n-2})$ and $\varepsilon_{s-1} + \varepsilon_s + t(\varepsilon_{s-1} + \varepsilon_s) = 2\varpi_n$.

- $t(\varepsilon_{s-1} - \varepsilon_s) = (\varepsilon_1 + \varepsilon_2) + \dots + (\varepsilon_{n-3} + \varepsilon_{n-2}) + 2\varepsilon_n$ and $\varepsilon_{s-1} - \varepsilon_s + t(\varepsilon_{s-1} - \varepsilon_s) = 2\varpi_n$.

- For $1 \leq i \leq s/2 - 1$:

(1) If $n \leq 4i - 2$,

$$\begin{aligned} t(\varepsilon_{2i-1} - \varepsilon_{2i}) &= 2 \sum_{j=1}^{n-2i} (\varepsilon_{n-j} - \varepsilon_j) + 4 \sum_{j=1}^{n/2-i} (\varepsilon_{2j-1} + \varepsilon_{2j}) \\ &\quad + 2 \sum_{j=n/2-i+1}^{i-1} (\varepsilon_{2j-1} + \varepsilon_{2j}) + (\varepsilon_{2i-1} + \varepsilon_{2i}) + 2\varepsilon_n \end{aligned}$$

$$\text{and } (\varepsilon_{2i-1} - \varepsilon_{2i}) + t(\varepsilon_{2i-1} - \varepsilon_{2i}) = 4\varpi_n.$$

(2) If $n > 4i - 2$,

$$\begin{aligned} t(\varepsilon_{2i-1} - \varepsilon_{2i}) &= 2 \sum_{j=1}^{2i-1} (\varepsilon_{n-j} - \varepsilon_j) + 4 \sum_{j=1}^{i-1} (\varepsilon_{2j-1} + \varepsilon_{2j}) \\ &\quad + 2 \sum_{j=i+1}^{n/2-i} (\varepsilon_{2j-1} + \varepsilon_{2j}) + 3(\varepsilon_{2i-1} + \varepsilon_{2i}) + 2\varepsilon_n \end{aligned}$$

$$\text{and } (\varepsilon_{2i-1} - \varepsilon_{2i}) + t(\varepsilon_{2i-1} - \varepsilon_{2i}) = 4\varpi_n.$$

Hence for all $1 \leq i \leq s/2 - 1$, $\varepsilon_{2i-1} - \varepsilon_{2i} + t(\varepsilon_{2i-1} - \varepsilon_{2i}) = 4\varpi_n$.

We conclude that when $n = s$ the product $\prod_{\gamma \in T} (1 - e^{-(\gamma + t(\gamma))})^{-1}$ is given by the right hand side of equality (1) of Section 4.6.1 and hence coincides with the lower bound for $\text{ch } Y(\mathfrak{p})$.

Now assume that $n > s$. The previous computations hold if we replace n by s and $2\varpi_n$ by ϖ_s (and so $4\varpi_n$ by $2\varpi_s$). Then we may recover $t(\gamma)$ and $\gamma + t(\gamma)$ for $\gamma = \varepsilon_{s-1} + \varepsilon_s$, $\gamma = \varepsilon_{s-1} - \varepsilon_s$ or $\gamma = \varepsilon_{2i-1} - \varepsilon_{2i}$, $1 \leq i \leq s/2 - 1$, by the above.

It remains to compute $t(\gamma)$, $\gamma + t(\gamma)$ for the rest of the elements in T .

- $t(\varepsilon_{s-1} - \varepsilon_{s+1}) = 2((\varepsilon_1 + \varepsilon_2) + \dots + (\varepsilon_{s-3} + \varepsilon_{s-2})) + (\varepsilon_{s-1} + \varepsilon_{s+1}) + 2\varepsilon_s$ and $(\varepsilon_{s-1} - \varepsilon_{s+1}) + t(\varepsilon_{s-1} - \varepsilon_{s+1}) = 2\varpi_s$.

- For $1 \leq j \leq [(n-s)/2]$,

$$\begin{aligned} t(-\varepsilon_{s+2j-1} + \varepsilon_{s+2j}) &= 2((\varepsilon_1 + \varepsilon_2) + \dots + (\varepsilon_{s-3} + \varepsilon_{s-2})) + 2(\varepsilon_{s-1} + \varepsilon_{s+1}) \\ &\quad - 2 \sum_{k=1}^{j-1} (\varepsilon_{s+2k-1} + \varepsilon_{s+2k}) + 2 \sum_{k=1}^{j-1} (\varepsilon_{s+2k} + \varepsilon_{s+2k+1}) \\ &\quad - (\varepsilon_{s+2j-1} + \varepsilon_{s+2j}) + 2\varepsilon_s \end{aligned}$$

$$\text{and } (-\varepsilon_{s+2j-1} + \varepsilon_{s+2j}) + t(-\varepsilon_{s+2j-1} + \varepsilon_{s+2j}) = 2\varpi_s.$$

- For $1 \leq j \leq [(n-s-1)/2]$,

$$\begin{aligned} t(\varepsilon_{s+2j} - \varepsilon_{s+2j+1}) &= 2((\varepsilon_1 + \varepsilon_2) + \dots + (\varepsilon_{s-3} + \varepsilon_{s-2})) + 2(\varepsilon_{s-1} + \varepsilon_{s+1}) \\ &\quad - 2 \sum_{k=1}^j (\varepsilon_{s+2k-1} + \varepsilon_{s+2k}) + 2 \sum_{k=1}^{j-1} (\varepsilon_{s+2k} + \varepsilon_{s+2k+1}) \\ &\quad + (\varepsilon_{s+2j} + \varepsilon_{s+2j+1}) + 2\varepsilon_s \end{aligned}$$

and $(\varepsilon_{s+2j} - \varepsilon_{s+2j+1}) + t(\varepsilon_{s+2j} - \varepsilon_{s+2j+1}) = 2\varpi_s$.

We conclude that also for $n > s$ the product $\prod_{\gamma \in T} (1 - e^{-(\gamma + t(\gamma))})^{-1}$ is given by the right hand side of equality (2) of Section 4.6.1 and hence coincides with the lower bound for $\text{ch } Y(\mathfrak{p})$.

4.6.3. *Conclusion.* Recall 2.9 and 2.10.

Theorem. Let \mathfrak{g} be a simple Lie algebra of type B_n , $n \geq 2$, and let $\mathfrak{p} = \mathfrak{p}_{\pi', \Lambda}^-$ be a truncated maximal parabolic subalgebra of \mathfrak{g} associated to $\pi' = \pi \setminus \{\alpha_s\}$, where s is an even integer, $s \leq n$.

There exists an adapted pair (h, y) for \mathfrak{p} and an affine slice $y + \mathfrak{g}_T$ in \mathfrak{p}^* such that restriction of functions gives an isomorphism of algebras between $Y(\mathfrak{p})$ and the ring $R[y + \mathfrak{g}_T]$ of polynomial functions on $y + \mathfrak{g}_T$.

In particular $Y(\mathfrak{p})$ is a polynomial algebra over k and the field $C(\mathfrak{p}_{\pi'}^-)$ of invariant fractions is a purely transcendental extension of k .

Remarks. (1) In the particular case $s = 2$ polynomiality was known by [21] and an adapted pair was constructed in [16]. Our adapted pair is equivalent to the adapted pair of Joseph $(h', y' = \sum_{s \in S'} x_s)$, in the sense of [7, 2.1.1]. Indeed one verifies that $w = \prod_{k=1}^{[(n-1)/2]} r_{\varepsilon_{2k+1}} \circ r_{\alpha_1} \in W_{\pi'}$ and sends bijectively S to S' .

(2) The degrees of a set of homogeneous generators of $Y(\mathfrak{p})$ are equal to the eigenvalues of $\text{ad } h$ on \mathfrak{g}_T computed in Section 4.5 each augmented by 1.

5. TYPE D, NON-EXTREMAL CASES

Let \mathfrak{g} be a simple Lie algebra of type D_n ($n \geq 4$) and consider the truncated maximal parabolic subalgebra $\mathfrak{p} = \mathfrak{p}_{\pi', \Lambda}^-$ associated to $\pi' = \pi \setminus \{\alpha_s\}$ with s even, $2 \leq s \leq n-2$. By [8, 5.1] the lower and upper bounds of 2.7 for $\text{ch } Y(\mathfrak{p})$ do not coincide. We will construct an adapted pair (h, y) for \mathfrak{p} and show that the algebra $Y(\mathfrak{p})$ is a polynomial algebra over k .

5.1. The set S . Let s be an even integer and $2 \leq s \leq n-2$. Let $\{\varepsilon_i\}_{1 \leq i \leq n}$ be an orthonormal basis for \mathbb{R}^n that is used to expand all simple roots α_i ($1 \leq i \leq n$) of π [1, Planche IV].

Recall the strongly orthogonal positive roots $\beta_i, \beta_{i'}, \beta_{i''}$ given in [8, Table I]. First we will give an erratum to [8, Table I], for D_n ; when n odd, we did not list

$\beta_{(n+1)/2} = \varepsilon_{n-2} - \varepsilon_{n-1} = \alpha_{n-2}$. Summarizing [8, Table I], for type D_n , we have the following strongly positive orthogonal roots :

- for all $1 \leq i \leq [n/2]$, $\beta_i = \varepsilon_{2i-1} + \varepsilon_{2i}$,
- for all $1 \leq i' \leq [n/2] - 1$, $\beta_{i'} = \varepsilon_{2i'-1} - \varepsilon_{2i'}$,
- if n is even, $\beta_{(\frac{n-2}{2})''} = \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n$
- if n is odd, $\beta_{(n+1)/2} = \alpha_{n-2} = \varepsilon_{n-2} - \varepsilon_{n-1}$.

Set $S = S^+ \sqcup S^- \sqcup S^m$ with

$$S^+ = \{\varepsilon_{2i-1} + \varepsilon_{2i}, \varepsilon_{s-1} + \varepsilon_{s+1}, \varepsilon_{2j} + \varepsilon_{2j+1} \mid 1 \leq i \leq s/2 - 1, s/2 + 1 \leq j \leq [(n-2)/2]\},$$

$$S^- = \{\varepsilon_{s-i} - \varepsilon_i, -\varepsilon_{2j-1} - \varepsilon_{2j} \mid 1 \leq i \leq s/2 - 1, s/2 + 1 \leq j \leq [(n-1)/2]\}$$

$$\text{and } S^m = \{\varepsilon_s - \varepsilon_n, \varepsilon_s + \varepsilon_n\}.$$

One has that $S \subset \Delta^+ \sqcup \Delta_{\pi'}^-$ and $|S| = n - 1 = \dim \mathfrak{h}_\Lambda$.

Observe that S in type D_n is almost identical with the set S in type B_n . We first show below that condition (1) of lemma 3.8 holds.

Lemma. $S|_{\mathfrak{h}_\Lambda}$ is a basis for \mathfrak{h}_Λ^* .

Proof. Set $S = \{s_i\}_{1 \leq i \leq n-1}$ with $s_{n-2} = \varepsilon_s - \varepsilon_n$ and $s_{n-1} = \varepsilon_s + \varepsilon_n$ and choose $\{h_i\}_{1 \leq i \leq n-1} = \{\alpha_i^\vee\}_{1 \leq i \leq n, i \neq s}$ as a basis of \mathfrak{h}_Λ .

Denote by $s'_{n-2} = \varepsilon_s$ and $s'_{n-1} = \varepsilon_n$ and $s'_i = s_i$ for all $1 \leq i \leq n-3$ and set $S' = \{s'_i\}_{1 \leq i \leq n-1}$. It is sufficient to prove that $\det(s'_i(h_j))_{1 \leq i, j \leq n-1} \neq 0$.

By ordering the basis of \mathfrak{h}_Λ as $\{\alpha_{2i}^\vee, \alpha_{s-1}^\vee, \alpha_{2j-1}^\vee, \alpha_{s-2j-1}^\vee, \alpha_k^\vee \mid 1 \leq i \leq s/2 - 1, 1 \leq j \leq [s/4], s+1 \leq k \leq n\}$ without repetitions and the elements of S' as $\{\beta_i, \varepsilon_s, \varepsilon_{s-i} - \varepsilon_i, \varepsilon_{s-1} + \varepsilon_{s+1}, -\varepsilon_{s+2j-1} - \varepsilon_{s+2j}, \varepsilon_{s+2j} + \varepsilon_{s+2j+1}, \varepsilon_n \mid 1 \leq i \leq s/2 - 1, 1 \leq j \leq (n-s-2)/2\}$ if n is even and $\{\beta_i, \varepsilon_s, \varepsilon_{s-i} - \varepsilon_i, \varepsilon_{s-1} + \varepsilon_{s+1}, -\varepsilon_{s+2j-1} - \varepsilon_{s+2j}, \varepsilon_{s+2j} + \varepsilon_{s+2j+1}, -\varepsilon_{n-2} - \varepsilon_{n-1}, \varepsilon_n \mid 1 \leq i \leq s/2 - 1, 1 \leq j \leq (n-s-3)/2\}$ if n is odd, one checks that

$$(s'_i(h_j))_{1 \leq i, j \leq n-1} = \begin{pmatrix} A & 0 & 0 \\ * & B & 0 \\ * & * & C \end{pmatrix}$$

where A (resp. B) is a $(s/2 - 1) \times (s/2 - 1)$ (resp. a $(s/2) \times (s/2)$) lower triangular matrix with 1 (resp. -1) on the diagonal. Moreover $C = \begin{pmatrix} C' & 0 \\ * & C'' \end{pmatrix}$ with C' an $(n-s-2) \times (n-s-2)$ lower triangular matrix with alternating 1 and -1 on the diagonal and C'' a 2×2 matrix. Then $\det(C') = (-1)^{[(n-s-2)/2]}$ and $\det(C'') = (-1)^{n-s} \times 2$.

We conclude that $\det(s'_i(h_j))_{1 \leq i, j \leq n-1} \neq 0$ hence the assertion of the lemma. \square

5.2. The Heisenberg sets. Recall section 5.1. For each $\gamma \in S$, we will define the Heisenberg set Γ_γ . Set $\Gamma^\pm = \bigsqcup_{\gamma \in S^\pm} \Gamma_\gamma$ and $\Gamma^m = \bigsqcup_{\gamma \in S^m} \Gamma_\gamma$.

5.2.1. *The set Γ^+ .* For all i , with $1 \leq i \leq s/2 - 1$, set $\Gamma_{\beta_i} = H_{\beta_i} \setminus \{\varepsilon_{2i-1} - \varepsilon_n, \varepsilon_{2i} + \varepsilon_n\}$ where H_{β_i} was defined in [8, Lemma 3].

Set $\Gamma_{\varepsilon_{s-1} + \varepsilon_{s+1}} = \{\varepsilon_{s-1} + \varepsilon_{s+1}, \varepsilon_{s-1} + \varepsilon_i, \varepsilon_{s+1} - \varepsilon_i, \varepsilon_{s-1} - \varepsilon_j, \varepsilon_{s+1} + \varepsilon_j \mid s+2 \leq i \leq n, s+2 \leq j \leq n-1\}$.

For all i , with $s/2 + 1 \leq i \leq [(n-2)/2]$, set

$\Gamma_{\varepsilon_{2i} + \varepsilon_{2i+1}} = \{\varepsilon_{2i} + \varepsilon_{2i+1}, \varepsilon_{2i} - \varepsilon_j, \varepsilon_j + \varepsilon_{2i+1}, \varepsilon_{2i} + \varepsilon_k, \varepsilon_{2i+1} - \varepsilon_k \mid 2i+2 \leq j \leq n, 2i+2 \leq k \leq n-1\}$.

5.2.2. *The set Γ^- .* For all i , with $1 \leq i \leq s/2 - 1$, set $\Gamma_{\varepsilon_{s-i} - \varepsilon_i} = \{\varepsilon_{s-i} - \varepsilon_i, \varepsilon_j - \varepsilon_i, \varepsilon_{s-i} - \varepsilon_j \mid i+1 \leq j \leq s-i-1\}$.

For all i , with $s/2 + 1 \leq i \leq [(n-1)/2]$, set

$\Gamma_{-\varepsilon_{2i-1} - \varepsilon_{2i}} = \{-\varepsilon_{2i-1} - \varepsilon_{2i}, -\varepsilon_{2i-1} - \varepsilon_j, \varepsilon_j - \varepsilon_{2i}, -\varepsilon_{2i-1} + \varepsilon_k, -\varepsilon_k - \varepsilon_{2i} \mid 2i+1 \leq j \leq n-1, 2i+1 \leq k \leq n\}$.

5.2.3. *The set Γ^m .* Set

$\Gamma_{\varepsilon_s - \varepsilon_n} = \{\varepsilon_s - \varepsilon_n, \varepsilon_s - \varepsilon_{2i-1}, \varepsilon_{2i-1} - \varepsilon_n, \varepsilon_s + \varepsilon_{2j+1}, -\varepsilon_{2j+1} - \varepsilon_n \mid 1 \leq i \leq [n/2], i \neq s/2 + 1, s/2 \leq j \leq [(n-2)/2]\}$ and

$\Gamma_{\varepsilon_s + \varepsilon_n} = \{\varepsilon_s + \varepsilon_n, \varepsilon_s - \varepsilon_{2i}, \varepsilon_{2i} + \varepsilon_n, \varepsilon_s - \varepsilon_{s+1}, \varepsilon_{s+1} + \varepsilon_n, \varepsilon_s + \varepsilon_{2j}, -\varepsilon_{2j} + \varepsilon_n \mid 1 \leq i \leq [(n-1)/2], i \neq s/2, s/2 + 1 \leq j \leq [(n-1)/2]\}$.

By construction, the sets Γ_γ , $\gamma \in S$, are disjoint Heisenberg sets of centre γ , included in $\Delta^+ \sqcup \Delta_{\pi'}^-$.

5.3. **Conditions (2), (3) and (4) of lemma 3.8.** Unlike type B , here some roots α in O such that $S_\alpha \cap O^m \neq \emptyset$ do not belong to $O_1 \sqcup O_2$, which makes type D more complicated.

Denote by π'_1 the connected component of π' of type A_{s-1} and by π'_2 the connected component of π' of type D_{n-s} (or $A_1 \times A_1$ if $s = n-2$).

5.3.1. Let $\alpha \in O^\pm$ and assume that there exists $\beta \in O^\pm \sqcup O^m$ such that $\alpha + \beta \in S$.

First case : $\alpha \in \Gamma_{\beta_i}^0 \subset O^+$ with $1 \leq i \leq s/2 - 1$. As in 4.3.1 one checks that, if $\beta \in O^+ \sqcup (O^m \cap \Delta^+)$, then $\beta = \theta(\alpha)$. Hence condition (2) for α . Assume now that $\beta \in O^m \cap \Delta^-$ which implies that $S_\alpha \cap O^m \neq \emptyset$. Then four cases occur : $\alpha = \varepsilon_{2i} - \varepsilon_n$ and $\beta = \varepsilon_s - \varepsilon_{2i} \in \Gamma_{\varepsilon_s + \varepsilon_n}^0$, or $\alpha = \varepsilon_{2i-1} + \varepsilon_n$ and $\beta = \varepsilon_s - \varepsilon_{2i-1} \in \Gamma_{\varepsilon_s - \varepsilon_n}^0$, or $\alpha = \varepsilon_{2i-1} - \varepsilon_s$ and $\beta = \varepsilon_s - \varepsilon_{s-2i+1}$ with $s-2i+1 < 2i-1$, or $\alpha = \varepsilon_{2i} - \varepsilon_s$ and $\beta = \varepsilon_s - \varepsilon_{s-2i}$ with $s-2i < 2i$.

Let consider just one of the two first cases, that is when $\alpha = \varepsilon_{2i-1} + \varepsilon_n$ and $\beta = \varepsilon_s - \varepsilon_{2i-1} \in \Gamma_{\varepsilon_s - \varepsilon_n}^0$. Then $\alpha + \beta = \varepsilon_s + \varepsilon_n$, $\theta(\alpha) = \varepsilon_{2i} - \varepsilon_n$ and $\theta(\beta) = \varepsilon_{2i-1} - \varepsilon_n$. One verifies that there exists $\gamma = \varepsilon_s - \varepsilon_{2i} \in \Gamma_{\varepsilon_s + \varepsilon_n}^0$ such that $\theta(\alpha) + \gamma = \varepsilon_s - \varepsilon_n$, $\theta(\beta) + \theta(\gamma) = \varepsilon_{2i-1} + \varepsilon_{2i}$ and that $\alpha, \theta(\alpha), \theta(\beta), \theta(\gamma) \in O_2$. If $i = 1$ or $s-2i+1 \leq 2i-1$ (resp. $s-2i \leq 2i$) then $\beta \in O_2$ (resp. $\gamma \in O_2$). Otherwise $\beta \in O_3$, $\tilde{\beta} = \varepsilon_{s-2i+1} - \varepsilon_s \in O_2 \cap S_\beta$ and $\theta(\tilde{\beta}) = \varepsilon_{s-2i+2} + \varepsilon_s \in O_1$ (resp. $\gamma \in O_3$, $\tilde{\gamma} = \varepsilon_{s-2i} - \varepsilon_s \in O_2 \cap S_\gamma$ and

$\theta(\tilde{\gamma}) = \varepsilon_{s-2i-1} + \varepsilon_s \in O_1$). Hence $\tilde{\beta}$ (resp. $\tilde{\gamma}$) satisfies the stationary condition. Thus $\alpha \in O_{cyc}$ and by remark (1) in 3.7 the roots $\beta, \gamma, \theta(\alpha), \theta(\beta), \theta(\gamma)$ are also cyclic roots.

Let consider just one of the two last cases. Suppose that $\alpha = \varepsilon_{2i-1} - \varepsilon_s$ and $\beta = \varepsilon_s - \varepsilon_{s-2i+1}$ with $s - 2i + 1 < 2i - 1$. By the above, $\beta \in O_{cyc} \cap O_3$ and $\tilde{\beta} = \alpha$. Hence condition (4) is satisfied for α .

Second case : $\alpha \in \Gamma_{\varepsilon_{s-1}+\varepsilon_{s+1}}^0 \subset O^+$. One easily checks that if $\beta \in O^+$ then $\beta = \theta(\alpha)$. Hence condition (2). Now if $\beta \in \Gamma_{\varepsilon_s-\varepsilon_n}^0$ then necessarily $\alpha = \varepsilon_{s-1} + \varepsilon_n$, $\beta = \varepsilon_s - \varepsilon_{s-1}$ and $\alpha + \beta = \varepsilon_s + \varepsilon_n$. One checks that $\gamma = \varepsilon_s - \varepsilon_{s+1} \in \Gamma_{\varepsilon_s+\varepsilon_n}^0$ verifies $\theta(\alpha) + \gamma = \varepsilon_s - \varepsilon_n$ and $\theta(\beta) + \theta(\gamma) = \varepsilon_{s-1} + \varepsilon_{s+1}$. Moreover $\alpha, \theta(\alpha), \beta, \theta(\beta), \gamma, \theta(\gamma) \in O_2$. Hence $\alpha \in O_{cyc}$. A similar computation shows that if $\beta \in \Gamma_{\varepsilon_s+\varepsilon_n}^0$, then $\alpha \in O_{cyc}$. Hence condition (4).

Third case : $\alpha \in \Gamma_{\varepsilon_{2i}+\varepsilon_{2i+1}}^0 \subset O^+$ with $s/2 + 1 \leq i \leq [(n-2)/2]$. One easily checks that if $\beta \in O^+$ then $\beta = \theta(\alpha)$. Hence condition (2).

Assume now that $\beta \in \Gamma_{\varepsilon_s-\varepsilon_n}^0$. Then necessarily $\alpha = \varepsilon_{2i+1} + \varepsilon_n$, $\beta = \varepsilon_s - \varepsilon_{2i+1}$ and $\alpha + \beta = \varepsilon_s + \varepsilon_n$. Then $\gamma = \varepsilon_s - \varepsilon_{2i} \in \Gamma_{\varepsilon_s+\varepsilon_n}^0$ verifies $\theta(\alpha) + \gamma = \varepsilon_s - \varepsilon_n$ and $\theta(\beta) + \theta(\gamma) = \varepsilon_{2i} + \varepsilon_{2i+1}$. Moreover one has that $\alpha, \theta(\alpha), \beta, \theta(\beta), \gamma, \theta(\gamma) \in O_2$. Hence $\alpha \in O_{cyc}$. A similar computation shows that if $\beta \in \Gamma_{\varepsilon_s+\varepsilon_n}^0$, then $\alpha \in O_{cyc}$. Hence condition (4).

Fourth case : $\alpha \in \Gamma_{\varepsilon_{s-i}-\varepsilon_i}^0 \subset O^-$ with $1 \leq i \leq s/2 - 1$. If $\beta \in \Gamma_{\varepsilon_{s-j}-\varepsilon_j}^0$ with $1 \leq j \leq s/2 - 1$ then one checks that $j = i$ and $\alpha + \beta = \varepsilon_{s-i} - \varepsilon_i$ thus $\beta = \theta(\alpha)$. Moreover one checks that it is not possible that $\beta \in \Gamma_{-\varepsilon_{2j-1}-\varepsilon_{2j}}^0$ with $s/2 + 1 \leq j \leq [(n-1)/2]$ since $\alpha \in \Delta_{\pi_1}^-$ whilst $\beta \in \Delta_{\pi_2}^-$ nor it is possible that $\beta \in O^m$. Hence condition (3) for α is satisfied.

Fifth case : $\alpha \in \Gamma_{-\varepsilon_{2i-1}-\varepsilon_{2i}}^0 \subset O^-$ with $s/2 + 1 \leq i \leq [(n-1)/2]$. If $\beta \in \Gamma_{-\varepsilon_{2j-1}-\varepsilon_{2j}}^0$ with $s/2 + 1 \leq j \leq [(n-1)/2]$ then one checks that $i = j$ and $\alpha + \beta = -\varepsilon_{2i-1} - \varepsilon_{2i}$, thus $\beta = \theta(\alpha)$. Hence condition (3).

Assume now that $\beta \in \Gamma_{\varepsilon_s-\varepsilon_n}^0$. Then necessarily $\alpha = -\varepsilon_{2i-1} + \varepsilon_n$ and $\beta = \varepsilon_s + \varepsilon_{2i-1}$. Moreover $\gamma = \varepsilon_s + \varepsilon_{2i} \in \Gamma_{\varepsilon_s+\varepsilon_n}^0$ verifies $\theta(\alpha) + \gamma = \varepsilon_s - \varepsilon_n$ and $\theta(\beta) + \theta(\gamma) = -\varepsilon_{2i-1} - \varepsilon_{2i}$. All these roots belong to O_2 . Hence $\alpha \in O_{cyc}$. A similar computation shows that if $\beta \in \Gamma_{\varepsilon_s+\varepsilon_n}^0$, then $\alpha \in O_{cyc}$. Hence condition (4).

5.3.2. Assume that $\alpha \in \Gamma_{\varepsilon_s-\varepsilon_n}^0 \subset O^m$. We show below that condition (4) holds.

First case : $\alpha = \varepsilon_s - \varepsilon_{2i-1}$ with $1 \leq i \leq s/2 - 1$. Then there exists $\beta = \varepsilon_{2i-1} + \varepsilon_n \in O^+$ such that $\alpha + \beta \in S$ and first case of 5.3.1, $\beta \in O_{cyc}$ and $\alpha \in O_{cyc}$. If now $\alpha = \varepsilon_{2i-1} - \varepsilon_n$ with $1 \leq i \leq s/2 - 1$ then $\alpha = \theta(\varepsilon_s - \varepsilon_{2i-1}) \in O_{cyc}$ (by remark (1) in 3.7) since $\varepsilon_s - \varepsilon_{2i-1} \in O_{cyc}$.

Second case : $\alpha = \varepsilon_s - \varepsilon_{s-1}$. By second case of 5.3.1, $\alpha \in O_{cyc}$. If now $\alpha = \varepsilon_{s-1} - \varepsilon_n$ then $\alpha = \theta(\varepsilon_s - \varepsilon_{s-1})$, thus $\alpha \in O_{cyc}$ by remark (1) in 3.7.

Third case : $\alpha = \varepsilon_s - \varepsilon_{2i-1}$ with $s/2 + 2 \leq i \leq [n/2]$. Then $\beta = \varepsilon_{2i-1} + \varepsilon_n \in \Gamma_{\varepsilon_{2i-2} + \varepsilon_{2i-1}}^0 \subset O^+$ is such (third case of 5.3.1) $\beta \in O_{cyc}$ and $\alpha \in C_\beta$, thus by remark (1) in 3.7 one has that $\alpha \in O_{cyc}$. If now $\alpha = \varepsilon_{2i-1} - \varepsilon_n$, with $s/2 + 2 \leq i \leq [n/2]$, then $\alpha = \theta(\varepsilon_s - \varepsilon_{2i-1})$ and by remark (1) in 3.7 one has that $\alpha \in O_{cyc}$.

Fourth case : $\alpha = \varepsilon_s + \varepsilon_{2i+1}$ with $s/2 \leq i \leq [(n-2)/2]$. If $i \neq [(n-2)/2]$ or n odd then $\beta = -\varepsilon_{2i+1} + \varepsilon_n \in \Gamma_{-\varepsilon_{2i+1} - \varepsilon_{2i+2}}^0 \subset O^-$ is such that (fifth case of 5.3.1) $\beta \in O_{cyc}$ and $\alpha \in C_\beta$ and then, by remark (1) in 3.7, $\alpha \in O_{cyc}$.

If $i = (n-2)/2$ and n even, then $\alpha = \varepsilon_s + \varepsilon_{n-1} \in O_1$ and $\theta(\alpha) = -\varepsilon_{n-1} - \varepsilon_n \in O_1$ then in this case $\alpha \in O_{st}$.

Now suppose that $\alpha = -\varepsilon_{2i+1} - \varepsilon_n$ with $s/2 \leq i \leq [(n-2)/2]$. If $i \neq [(n-2)/2]$ or n odd, then $\alpha = \theta(\varepsilon_s + \varepsilon_{2i+1})$, hence by remark (1) in 3.7, $\alpha \in O_{cyc}$.

Finally if $\alpha = -\varepsilon_{n-1} - \varepsilon_n$ (that is, when n is even) then $\alpha = \theta(\varepsilon_s + \varepsilon_{n-1}) \in O_{st}$ (since $\theta(\alpha) \in O_{st}$).

Similar computations may be done for $\alpha \in \Gamma_{\varepsilon_s + \varepsilon_n}^0$; we leave the details as an exercise. We conclude that conditions (2)-(3)-(4) of lemma 3.8 hold.

5.4. The set T . Let T denote the complement of the set $\Gamma = \Gamma^+ \sqcup \Gamma^- \sqcup \Gamma^m$ in $\Delta^+ \sqcup \Delta_{\pi'}^-$.

One checks that $T = \{\varepsilon_{s-1} + \varepsilon_s, \varepsilon_{s-1} - \varepsilon_{s+1}, \varepsilon_{2i-1} - \varepsilon_{2i}, \varepsilon_{2j} - \varepsilon_{2j+1}, -\varepsilon_{2k+1} + \varepsilon_{2k+2} \mid 1 \leq i \leq s/2, s/2 + 1 \leq j \leq [(n-1)/2], s/2 \leq k \leq [(n-2)/2]\}$. Comparing with Section 4.4 we see that it coincides with the set T in type B_n for the same s . Hence $|T| = n - s/2 + 1$. Moreover the $\langle \mathbf{ij} \rangle$ -orbits are the same as in type B_n hence in $n - s/2 + 1$ in number. Thus $|T| = \text{ind } \mathfrak{p}_{\pi', \Lambda}$.

Remark. All conditions of lemma 3.1 are satisfied. Hence defining $h \in \mathfrak{h}_\Lambda$ by $\gamma(h) = -1$ for all $\gamma \in S$, and setting $y = \sum_{\gamma \in S} x_\gamma$ we obtain an adapted pair (h, y) for $\mathfrak{p}_{\pi', \Lambda}^-$.

5.5. The eigenvalues of $\text{ad } h$ on \mathfrak{g}_T . As in type B, we give an expansion of the semisimple element h :

$$\begin{aligned} h = & \sum_{k=1}^{[s/4]} \left(\frac{s}{2} + 2k - 1 \right) \varepsilon_{2k-1} + \sum_{k=[s/4]+1}^{s/2-1} \left(\frac{3s}{2} - 2k \right) \varepsilon_{2k-1} \\ & - \sum_{k=1}^{[s/4]} \left(\frac{s}{2} + 2k \right) \varepsilon_{2k} - \sum_{k=[s/4]+1}^{s/2-1} \left(\frac{3s}{2} + 1 - 2k \right) \varepsilon_{2k} + \frac{s}{2} \varepsilon_{s-1} - \varepsilon_s \\ & + \sum_{k=1}^{[(n-s)/2]} \left(-2k + 1 - \frac{s}{2} \right) \varepsilon_{s+2k-1} + \sum_{k=1}^{[(n-s-1)/2]} \left(2k + \frac{s}{2} \right) \varepsilon_{s+2k}. \end{aligned}$$

The eigenvalues of $\text{ad } h$ on \mathfrak{g}_T are :

- $s + 4i - 1 = (\varepsilon_{2i-1} - \varepsilon_{2i})(h)$ for all i , with $1 \leq i \leq [s/4]$.
- $3s - 4i + 1 = (\varepsilon_{2i-1} - \varepsilon_{2i})(h)$ for all i , with $[s/4 + 1] \leq i \leq s/2 - 1$.
- $s/2 + 1 = (\varepsilon_{s-1} - \varepsilon_s)(h)$.
- $s/2 - 1 = (\varepsilon_{s-1} + \varepsilon_s)(h)$.

- $n - s/2 - 1 = \begin{cases} (-\varepsilon_{n-1} + \varepsilon_n)(h) & \text{if } n \text{ even.} \\ (\varepsilon_{n-1} - \varepsilon_n)(h) & \text{if } n \text{ odd.} \end{cases}$
- $s + 1 = (\varepsilon_{s-1} - \varepsilon_{s+1})(h)$.
- $s + 4j - 1 = (-\varepsilon_{s+2j-1} + \varepsilon_{s+2j})(h)$ for all j , with $1 \leq j \leq [(n - s - 1)/2]$.
- $s + 4j + 1 = (\varepsilon_{s+2j} - \varepsilon_{s+2j+1})(h)$ for all j , with $1 \leq j \leq [(n - s - 2)/2]$.

From the last three equalities we have that $s + 2k - 1$ is an eigenvalue of $\text{ad } h$ on \mathfrak{g}_T , for all k , with $1 \leq k \leq n - s - 1$.

5.6. Polynomiality of $Y(\mathfrak{p})$.

5.6.1. *The lower bound for $\text{ch } Y(\mathfrak{p})$.* Recall that the lower bound for $\text{ch } Y(\mathfrak{p})$ is :

$$\text{ch } Y(\mathfrak{p}) \geq \prod_{\Gamma \in E(\pi')} (1 - e^{\delta_\Gamma})^{-1}$$

The computation of the δ_Γ , $\Gamma \in E(\pi')$, is exactly as in 4.6.1, except for the $\langle \mathbf{ij} \rangle$ -orbit $\Gamma_{n-1} = \{\alpha_{n-1}\}$, for which $\delta_{\Gamma_{n-1}} = -2(\varpi_{n-1} - \varpi'_{n-1}) = -\varpi_s$.

Then the lower bound is equal to

$$\prod_{\Gamma \in E(\pi')} (1 - e^{\delta_\Gamma})^{-1} = (1 - e^{-\varpi_s})^{-3} (1 - e^{-2\varpi_s})^{-(n-2-s/2)}. \quad (3)$$

5.6.2. *The improved upper bound for $\text{ch } Y(\mathfrak{p})$.* We will now compute the improved upper bound (see 2.9) for $\text{ch } Y(\mathfrak{p})$ and as in type B_n , we will show that it is equal to the lower bound.

With the notations of Section 2.9 we have

$$\text{ch } Y(\mathfrak{p}) \leq \prod_{\gamma \in T} (1 - e^{-(\gamma + t(\gamma))})^{-1},$$

where the set T is given in Section 5.4.

Again the computations are very similar to type B of Section 4.6.2. If we compare the sets S (4.1 and 5.1) and the sets T (4.4 and 5.4) for type B_n and D_n with the same s , $2 \leq s \leq n - 2$, the sets T are identical and the sets S^\pm differ only by one element.

More precisely, if n is odd, then $\varepsilon_{n-1} + \varepsilon_n \notin S$, and so in this case for $\gamma = \varepsilon_{n-1} - \varepsilon_n \in T$, the element $t(\gamma)$ computed in 4.6.2 is no longer in $\mathbb{Q}S$. On the other hand,

$$\begin{aligned} t(\varepsilon_{n-1} - \varepsilon_n) &= (\varepsilon_1 + \varepsilon_2) + \dots + (\varepsilon_{s-3} + \varepsilon_{s-2}) + (\varepsilon_{s-1} + \varepsilon_{s+1}) \\ &\quad - \sum_{j=s/2+1}^{(n-1)/2} (\varepsilon_{2j-1} + \varepsilon_{2j}) + \sum_{j=s/2+1}^{(n-3)/2} (\varepsilon_{2j} + \varepsilon_{2j+1}) + (\varepsilon_s + \varepsilon_n) \in \mathbb{Q}S \end{aligned}$$

and $t(\varepsilon_{n-1} - \varepsilon_n) + (\varepsilon_{n-1} - \varepsilon_n) = \varpi_s$.

Similarly, if n is even, then $-(\varepsilon_{n-1} + \varepsilon_n) \notin S$ and for $\gamma = -\varepsilon_{n-1} + \varepsilon_n \in T$ one has that

$$\begin{aligned} t(-\varepsilon_{n-1} + \varepsilon_n) &= (\varepsilon_1 + \varepsilon_2) + \dots + (\varepsilon_{s-3} + \varepsilon_{s-2}) + (\varepsilon_{s-1} + \varepsilon_{s+1}) \\ &- \sum_{j=s/2+1}^{(n-2)/2} (\varepsilon_{2j-1} + \varepsilon_{2j}) + \sum_{j=s/2+1}^{(n-2)/2} (\varepsilon_{2j} + \varepsilon_{2j+1}) + (\varepsilon_s - \varepsilon_n) \in \mathbb{Q}S \end{aligned}$$

and $t(-\varepsilon_{n-1} + \varepsilon_n) + (-\varepsilon_{n-1} + \varepsilon_n) = \varpi_s$.

Note that in type B the corresponding weights $\gamma + t(\gamma)$ are equal to $2\varpi_s$ instead of ϖ_s , hence the improved upper bound for $Y(\mathfrak{p})$ in 4.6.2 will differ from the improved upper bound for D only by this factor. We conclude that the improved upper bound for $\text{ch } Y(\mathfrak{p})$ is equal to the lower bound.

5.6.3. Conclusion. Recall 2.9 and 2.10. From the result of the previous Section we deduce the following theorem.

Theorem. *Let \mathfrak{g} be a simple Lie algebra of type D_n ($n \geq 4$) and let $\mathfrak{p} = \mathfrak{p}_{\pi', \Lambda}^-$ be the truncated maximal parabolic subalgebra of \mathfrak{g} associated to $\pi' = \pi \setminus \{\alpha_s\}$, where s is an even integer with $s \leq n - 2$.*

There exists an adapted pair (h, y) for \mathfrak{p} and an affine slice $y + \mathfrak{g}_T$ in \mathfrak{p}^ such that restriction of functions gives an isomorphism of algebras between $Y(\mathfrak{p})$ and the ring $R[y + \mathfrak{g}_T]$ of polynomial functions on $y + \mathfrak{g}_T$.*

In particular $Y(\mathfrak{p})$ is a polynomial algebra over k and the field $C(\mathfrak{p}_{\pi'}^-)$ of invariant fractions is a purely transcendental extension of k .

Remarks. (1) When $s = 2$, the above result was known in [21] and was proven again by a different method in [16], where an adapted pair $(h', y' = \sum_{\gamma \in S'} x_\gamma)$ was constructed. Our adapted pair $(h, y = \sum_{\gamma \in S} x_\gamma)$ does not coincide with (h', y') but it is equivalent to it. Indeed, setting $r_{i,j} = r_{\varepsilon_i - \varepsilon_j} \circ r_{\varepsilon_i + \varepsilon_j}$, one verifies that $w = \prod_{k=1}^{2m-1} r_{2k+1, 2k+3} \circ r_{\alpha_1}$ (resp. $w = \prod_{k=1}^{2m-3} r_{2k+1, 2k+3} \circ r_{\alpha_1} \circ r_{n-1, n}$) if $n = 4m + u$ with $u \in \{1, 2, 3\}$ (resp. if $n = 4m$) and $m \neq 0$ is such that $w \in W_{\pi'}$ and sends bijectively S to S' .

(2) The degrees of a set of homogeneous generators of $Y(\mathfrak{p})$ are the eigenvalues of $\text{ad } h$ on \mathfrak{g}_T given in 5.5 each augmented by one.

6. TYPE D, EXTREMAL CASES

In this section, we assume that the simple Lie algebra \mathfrak{g} is of type D_n with $n \geq 6$ and n even. We consider $\mathfrak{p} = \mathfrak{p}_{\pi', \Lambda}^-$ the truncated maximal parabolic subalgebra of \mathfrak{g} associated to $\pi' = \pi \setminus \{\alpha_n\}$. Then the lower and the upper bounds of 2.7 for $\text{ch } Y(\mathfrak{p})$ do not coincide. We will construct an adapted pair for \mathfrak{p} and then prove that $Y(\mathfrak{p})$ is a polynomial algebra over k . Since the case when $\pi' = \pi \setminus \{\alpha_{n-1}\}$ is symmetric, this will also prove that $Y(\mathfrak{p})$ is polynomial when $\pi' = \pi \setminus \{\alpha_{n-1}\}$.

6.1. The set S . We set $S = \{\varepsilon_{2i-1} + \varepsilon_{2i}, \varepsilon_{n-3} + \varepsilon_{n-1}, \varepsilon_n - \varepsilon_{n-3}, \varepsilon_{n-2} - \varepsilon_{n-4}, \varepsilon_{n-4} - \varepsilon_{n-5}, \varepsilon_{n-3} - \varepsilon_{n-6}, \varepsilon_{n-2j} - \varepsilon_{n-2j-2} \mid 1 \leq i \leq n/2 - 2, 3 \leq j \leq n/2 - 2\}$. One checks that $S \subset \Delta^+ \sqcup \Delta_{\pi'}^-$ and that $|S| = n - 1 = \dim \mathfrak{h}_\Lambda$.

We first prove below that condition (1) of lemma 3.8 holds.

Lemma. $S|_{\mathfrak{h}_\Lambda}$ is a basis for \mathfrak{h}_Λ^* .

Proof. Set $S = \{s_i\}_{1 \leq i \leq n-1}$ with $s_i = \varepsilon_{2i-1} + \varepsilon_{2i}$ for all $i \in \mathbb{N}$, $1 \leq i \leq n/2 - 2$, $s_{n/2-1} = \varepsilon_{n-3} + \varepsilon_{n-1}$, $s_{n/2} = \varepsilon_{n-4} - \varepsilon_{n-5}$, $s_{n/2+1} = \varepsilon_{n-2} - \varepsilon_{n-4}$, $s_{n/2+2} = \varepsilon_n - \varepsilon_{n-3}$, $s_{n/2+3} = \varepsilon_{n-3} - \varepsilon_{n-6}$, $s_{n/2+k} = \varepsilon_{n-2k+2} - \varepsilon_{n-2k}$ for all $k \in \mathbb{N}$, with $4 \leq k \leq n/2 - 1$. Then set $s'_i = s_i$ for all $i \in \mathbb{N}$ with $1 \leq i \leq n - 1$, $i \neq n/2 - 1$, $s'_{n/2-1} = \varepsilon_{n-1} + \varepsilon_n = s_{n/2-1} + s_{n/2+2}$ and $S' = \{s'_i\}_{1 \leq i \leq n-1}$.

If we choose $\{h_i\}_{1 \leq i \leq n-1} = \{\alpha_i^\vee\}_{1 \leq i \leq n-1}$ as a basis of \mathfrak{h}_Λ , it is sufficient to show that $\det(s'_i(h_j))_{1 \leq i, j \leq n-1} \neq 0$.

By ordering S' as above and the basis of \mathfrak{h}_Λ as $\{\alpha_{2i}^\vee, \alpha_{n-5}^\vee, \alpha_{n-3}^\vee, \alpha_{n-1}^\vee, \alpha_{n-2j-1}^\vee \mid 1 \leq i \leq n/2 - 1, 3 \leq j \leq n/2 - 1\}$, one checks that the matrix $(s'_i(h_j))_{1 \leq i, j \leq n-1}$ is a lower triangular matrix with 1 on the first $n/2 - 2$ diagonal elements, then -1 , -2 , -1 , -1 on the next diagonal elements and then 1 on the $n/2 - 3$ last diagonal elements. Hence $\det(s'_i(h_j))_{1 \leq i, j \leq n-1} = 2$ and the lemma. \square

6.2. The Heisenberg sets. For all $k \in \mathbb{N}$, with $2 \leq k \leq n/2 - 3$, set $\Gamma_{\varepsilon_{2k} - \varepsilon_{2k-2}} = \{\varepsilon_{2k} - \varepsilon_{2k-2}, \varepsilon_{2k} - \varepsilon_i, \varepsilon_i - \varepsilon_{2k-2} \mid 1 \leq i \leq 2k - 3\}$.

Set $\Gamma_{\varepsilon_{n-3} - \varepsilon_{n-6}} = \{\varepsilon_{n-3} - \varepsilon_{n-6}, \varepsilon_{n-3} - \varepsilon_i, \varepsilon_i - \varepsilon_{n-6} \mid 1 \leq i \leq n - 7\}$,

$\Gamma_{\varepsilon_{n-4} - \varepsilon_{n-5}} = \{\varepsilon_{n-4} - \varepsilon_{n-5}, \varepsilon_{n-3} - \varepsilon_{n-5}, \varepsilon_{n-4} - \varepsilon_{n-3}, \varepsilon_{n-4} - \varepsilon_{2i}, \varepsilon_{2i} - \varepsilon_{n-5} \mid 1 \leq i \leq n/2 - 3\}$,

$\Gamma_{\varepsilon_{n-2} - \varepsilon_{n-4}} = \{\varepsilon_{n-2} - \varepsilon_{n-4}, \varepsilon_{n-2} - \varepsilon_{n-1}, \varepsilon_{n-1} - \varepsilon_{n-4}, \varepsilon_{n-2} - \varepsilon_n, \varepsilon_n - \varepsilon_{n-4}, \varepsilon_{n-2} - \varepsilon_i, \varepsilon_i - \varepsilon_{n-4} \mid 1 \leq i \leq n - 5\}$,

$\Gamma_{\varepsilon_n - \varepsilon_{n-3}} = \{\varepsilon_n - \varepsilon_{n-3}, \varepsilon_n - \varepsilon_{n-2}, \varepsilon_{n-2} - \varepsilon_{n-3}, \varepsilon_n - \varepsilon_{n-1}, \varepsilon_{n-1} - \varepsilon_{n-3}, \varepsilon_n - \varepsilon_i, \varepsilon_i - \varepsilon_{n-3} \mid 1 \leq i \leq n - 6\}$,

$\Gamma_{\varepsilon_{n-3} + \varepsilon_{n-1}} = \{\varepsilon_{n-3} + \varepsilon_{n-1}, \varepsilon_{n-3} + \varepsilon_n, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_{n-3} - \varepsilon_n, \varepsilon_n + \varepsilon_{n-1}, \varepsilon_{n-3} - \varepsilon_{n-2}, \varepsilon_{n-2} + \varepsilon_{n-1}, \varepsilon_{n-3} + \varepsilon_{n-2}, -\varepsilon_{n-2} + \varepsilon_{n-1}, \varepsilon_{n-1} - \varepsilon_i, \varepsilon_i + \varepsilon_{n-3} \mid 1 \leq i \leq n - 5\}$.

The $n/2 + 1$ sets defined above are Heisenberg sets, which we denote by Γ_{γ_j} , $1 \leq j \leq n/2 + 1$, whose centre will be denoted by $\gamma_j \in S$.

Let $i \in \mathbb{N}$, with $1 \leq i \leq n/2 - 2$, and recall that $\beta_i = \varepsilon_{2i-1} + \varepsilon_{2i}$ is a strongly orthogonal positive root of \mathfrak{g} (5.1) and that we denote by H_{β_i} the maximal Heisenberg set in Δ^+ of centre β_i ([8, Lemma 3]).

We define below every Heisenberg set Γ_{β_i} of centre β_i , $1 \leq i \leq n/2 - 2$, by decreasing induction on i .

First we set $\Gamma_{\beta_{n/2-2}} = (H_{\beta_{n/2-2}} \setminus \bigsqcup_{1 \leq j \leq n/2+1} \Gamma_{\gamma_j} \cap H_{\beta_{n/2-2}}) \sqcup \{\varepsilon_i + \varepsilon_{n-4}, \varepsilon_{n-5} - \varepsilon_i \mid 1 \leq i \leq n - 6\} \sqcup \{\varepsilon_{n-4} - \varepsilon_{2i-1}, \varepsilon_{2i-1} + \varepsilon_{n-5} \mid 1 \leq i \leq n/2 - 3\}$. Set $\gamma_j = \beta_{n-j}$ for all

$j \in \mathbb{N}$, with $n/2 + 2 \leq j \leq n - 1$ and suppose, for $2 \leq k \leq n/2 - 2$, that we have defined the Heisenberg set Γ_{γ_j} of centre $\gamma_j \in S$, for all $j \in \mathbb{N}$, with $1 \leq j \leq n/2 + k$.

Then we set $\Gamma_{\gamma_{n/2+k+1}} = \Gamma_{\beta_{n/2-k-1}} = (H_{\beta_{n/2-k-1}} \setminus \bigsqcup_{1 \leq j \leq n/2+k} \Gamma_{\gamma_j} \cap H_{\beta_{n/2-k-1}}) \sqcup \{\varepsilon_i + \varepsilon_{n-2k-2}, \varepsilon_{n-2k-3} - \varepsilon_i \mid 1 \leq i \leq n - 2k - 4\}$.

One checks that, for every $\gamma_j \in S$, $1 \leq j \leq n - 1$, the set Γ_{γ_j} is an Heisenberg set of centre γ_j . Moreover by construction all these Heisenberg sets are disjoint and $\Gamma = \bigsqcup_{\gamma \in S} \Gamma_{\gamma} \subset \Delta^+ \sqcup \Delta_{\pi'}^-$.

6.3. Conditions (2), (3) and (4) of lemma 3.8. Recall the definition of Γ^\pm and of Γ^m (3.8) and observe that, for $n \geq 8$, $\Gamma_{\beta_1} = \Gamma^+$ (that is, $\{\beta_1 = \varepsilon_1 + \varepsilon_2\} = S^+$) and $\Gamma^m = \bigsqcup_{\gamma \in S \setminus \{\beta_1\}} \Gamma_{\gamma}$ (that is, $S \setminus \{\beta_1\} = S^m$) and, for $n = 6$, $\Gamma_{\beta_1} = \Gamma^+$, $\Gamma_{\varepsilon_6 - \varepsilon_3} = \Gamma^-$ (here, $\{\varepsilon_6 - \varepsilon_3\} = S^-$) and $\Gamma^m = \bigsqcup_{\gamma \in S \setminus \{\beta_1, \varepsilon_6 - \varepsilon_3\}} \Gamma_{\gamma}$ (here, $S \setminus \{\beta_1, \varepsilon_6 - \varepsilon_3\} = S^m$).

In these extremal cases, we will see that conditions (2)-(3)-(4) of lemma 3.8 are more complicated to check than in non-extremal cases in type D since there are more roots in O which do not belong to $O_1 \sqcup O_2$. We will show below that conditions (2)-(3)-(4) of lemma 3.8 hold.

(a) The roots in $\Gamma_{\varepsilon_{2k} - \varepsilon_{2k-2}}^0$ for $2 \leq k \leq n/2 - 3$.

• Let $\alpha = \varepsilon_{2k} - \varepsilon_i \in \Gamma_{\varepsilon_{2k} - \varepsilon_{2k-2}}^0$ for $1 \leq i \leq 2k - 3$.

Assume first that i is even. Then one checks that α and $\theta(\alpha) = \varepsilon_i - \varepsilon_{2k-2}$ belong to O_3 (unless $i = 2$, in which case $\theta(\alpha) \in O_2$, $\alpha^1 = \varepsilon_1 + \varepsilon_{2k-2}$ and $\theta(\alpha^1) \in O_1$, hence $\alpha^2 = \alpha^1$ or $2k = n - 6$, in which case $\alpha \in O_2$, $\alpha^{(1)} = \varepsilon_{n-7} + \varepsilon_i$ and $\theta(\alpha^{(1)}) \in O_1$, hence $\alpha^{(2)} = \alpha^{(1)}$). Moreover when α , resp. $\theta(\alpha)$, belongs to O_3 then it satisfies the stability condition. Indeed, with the notations of 3.4, $\alpha' = \varepsilon_{2k-1} + \varepsilon_i \in \Gamma_{\beta_{i/2}}^0 \cap O_2$, $\theta(\alpha') = \varepsilon_{i-1} - \varepsilon_{2k-1} \in O_1$. Similarly $\theta(\alpha)' = \varepsilon_{i-1} + \varepsilon_{2k-2} \in \Gamma_{\beta_{k-1}}^0 \cap O_2$ and $\theta(\theta(\alpha)') = \varepsilon_{2k-3} - \varepsilon_{i-1} \in O_1$. On the other hand if $2k \leq n - 8$, then $\alpha^{(1)} = \theta(\alpha)^1 = \varepsilon_{i+2} - \varepsilon_{2k} \in \Gamma_{\varepsilon_{2k+2} - \varepsilon_{2k}}^0$ and if $i \geq 4$, $\alpha^1 = \varepsilon_{2k-2} - \varepsilon_{i-2} \in \Gamma_{\varepsilon_{2k-2} - \varepsilon_{2k-4}}^0$. One deduces by induction that $\alpha^{(1)}$ and α^1 satisfy the stationary condition, and then by remark (3) of 3.5, $\alpha \in O_{st}$.

Assume now that i is odd. Then one checks that α and $\theta(\alpha)$ belong to O_2 (unless $i = 2k - 3$, in which case $\theta(\alpha) \in O_1$, hence $\alpha^1 = \alpha$). One has that $\alpha^{(1)} = \varepsilon_{2k-1} + \varepsilon_i \in \Gamma_{\beta_{(i+1)/2}}^0 \cap O_2$, $\theta(\alpha^{(1)}) = \varepsilon_{i+1} - \varepsilon_{2k-1} \in O_2$ (unless $i = 1$, in which case $\theta(\alpha^{(1)}) \in O_1$, hence $\alpha^{(2)} = \alpha^{(1)}$), $\alpha^{(2)} = \varepsilon_{2k-1} - \varepsilon_{i-1} \in \Gamma_{\beta_k}^0 \cap O_2$, $\theta(\alpha^{(2)}) = \varepsilon_{2k} + \varepsilon_{i-1} \in O_2$.

Assume that $2k = n - 6$ (and $i \geq 3$). Then $\alpha^{(3)} = \varepsilon_{i-2} - \varepsilon_{n-6} \in \Gamma_{\varepsilon_{n-3} - \varepsilon_{n-6}}^0 \cap O_2$, $\theta(\alpha^{(3)}) = \varepsilon_{n-3} - \varepsilon_{i-2} \in O_2$, $\alpha^{(4)} = \varepsilon_{n-1} + \varepsilon_{i-2} \in \Gamma_{\beta_{(i-1)/2}}^0 \cap O_2$, $\theta(\alpha^{(4)}) = \varepsilon_{i-1} - \varepsilon_{n-1}$. If $i = 3$, then $\theta(\alpha^{(4)}) \in O_1$ and $\alpha^{(5)} = \alpha^{(4)}$. If $i \geq 5$, then $\theta(\alpha^{(4)}) \in O_2$, $\alpha^{(5)} = \varepsilon_{n-1} - \varepsilon_{i-3} \in \Gamma_{\varepsilon_{n-3} + \varepsilon_{n-1}}^0 \cap O_2$, $\theta(\alpha^{(5)}) = \varepsilon_{n-3} + \varepsilon_{i-3} \in O_1$ and $\alpha^{(6)} = \alpha^{(5)}$.

Now if $2k \leq n-8$ (and $i \geq 3$), then $\alpha^{(3)} = \varepsilon_{i-2} - \varepsilon_{2k} \in \Gamma_{\varepsilon_{2k+2}-\varepsilon_{2k}}^0$ and one deduces by induction that $\alpha^{(3)}$ satisfies the stationary condition.

On the other hand if $i \leq 2k-5$, then one has that $\alpha^1 = \varepsilon_{i+1} + \varepsilon_{2k-2} \in \Gamma_{\beta_{k-1}}^0 \cap O_2$, $\theta(\alpha^1) = \varepsilon_{2k-3} - \varepsilon_{i+1} \in O_2$ (unless $i = 2k-5$, in which case $\theta(\alpha^1) \in O_1$ and $\alpha^2 = \alpha^1$), $\alpha^2 = \varepsilon_{i+3} - \varepsilon_{2k-3} \in \Gamma_{\beta_{(i+3)/2}}^0 \cap O_2$, $\theta(\alpha^2) = \varepsilon_{i+2} + \varepsilon_{2k-3} \in O_2$, $\alpha^3 = \varepsilon_{2k-2} - \varepsilon_{i+2} \in \Gamma_{\varepsilon_{2k-2}-\varepsilon_{2k-4}}^0$ and one deduces by induction that α^3 satisfies the stationary condition. Hence $\alpha \in O_{st}$.

If now $\alpha = \varepsilon_i - \varepsilon_{2k-2}$, one has that $\alpha \in O_{st}$ since $\theta(\alpha) \in O_{st}$ by the above.

Hence condition (4) of lemma 3.8 is satisfied for all roots in $\Gamma_{\varepsilon_{2k}-\varepsilon_{2k-2}}^0$, $2 \leq k \leq n/2-3$.

(b) The roots in $\Gamma_{\varepsilon_{n-3}-\varepsilon_{n-6}}^0$.

• Let $\alpha = \varepsilon_{n-3} - \varepsilon_i \in \Gamma_{\varepsilon_{n-3}-\varepsilon_{n-6}}^0$, for $1 \leq i \leq n-7$.

Assume first that i is even. Then one checks that α and $\theta(\alpha) = \varepsilon_i - \varepsilon_{n-6}$ belong to O_3 and satisfy the stability condition. Indeed $\alpha' = \varepsilon_{n-1} + \varepsilon_i \in \Gamma_{\beta_{i/2}}^0 \cap O_2$, $\theta(\alpha') = \varepsilon_{i-1} - \varepsilon_{n-1} \in O_1$, $\theta(\alpha)' = \varepsilon_{i-1} + \varepsilon_{n-6} \in \Gamma_{\beta_{n/2-3}}^0 \cap O_2$ and $\theta(\theta(\alpha)') = \varepsilon_{n-7} - \varepsilon_{i-1} \in O_1$.

On the other hand $\alpha^{(1)} = \varepsilon_{i+2} - \varepsilon_{n-3} \in \Gamma_{\varepsilon_n-\varepsilon_{n-3}}^0 \cap O_2$ and $\alpha^1 = \varepsilon_{n-6} - \varepsilon_{i-2} \in \Gamma_{\varepsilon_{n-6}-\varepsilon_{n-8}}^0$.

By paragraph (a) above one has that $\alpha^1 \in O_{st}$. Moreover one has that $\theta(\alpha^{(1)}) = \varepsilon_n - \varepsilon_{i+2} \in O_2$, $\alpha^{(2)} = \varepsilon_{i+4} - \varepsilon_n \in \Gamma_{\beta_{(i+4)/2}}^0 \cap O_2$ and $\theta(\alpha^{(2)}) = \varepsilon_{i+3} + \varepsilon_n \in O_1$. Hence the root $\theta(\alpha)$ satisfies the stationary condition. By remark (4) of 3.6 one deduces that $\alpha \in O_{st}$.

Assume now that i is odd. Then one checks that α and $\theta(\alpha)$ belong to O_2 (unless $i = n-7$, in which case $\theta(\alpha) \in O_1$, hence α satisfies the stationary condition). One has that $\alpha^{(1)} = \varepsilon_i + \varepsilon_{n-1} \in \Gamma_{\beta_{(i+1)/2}}^0 \cap O_2$, $\theta(\alpha^{(1)}) = \varepsilon_{i+1} - \varepsilon_{n-1} \in O_2$ (unless $i = 1$, in which case $\theta(\alpha^{(1)}) \in O_1$, hence $\theta(\alpha)$ satisfies the stationary condition) and if $i \geq 3$, then $\alpha^{(2)} = \varepsilon_{n-1} - \varepsilon_{i-1} \in \Gamma_{\varepsilon_{n-3}+\varepsilon_{n-1}}^0 \cap O_2$, $\theta(\alpha^{(2)}) = \varepsilon_{n-3} + \varepsilon_{i-1} \in O_2$, $\alpha^{(3)} = \varepsilon_{i-2} - \varepsilon_{n-3} \in \Gamma_{\varepsilon_n-\varepsilon_{n-3}}^0 \cap O_2$ and $\theta(\alpha^{(3)}) = \varepsilon_n - \varepsilon_{i-2} \in O_1$. Hence the root $\theta(\alpha)$ satisfies the stationary condition.

On the other hand one has, for $i \leq n-9$, that $\alpha^1 = \varepsilon_{i+1} + \varepsilon_{n-6} \in \Gamma_{\beta_{n/2-3}}^0 \cap O_2$, $\theta(\alpha^1) = \varepsilon_{n-7} - \varepsilon_{i+1} \in O_2$, (unless $i = n-9$, in which case $\theta(\alpha^1) \in O_1$, hence α satisfies the stationary condition). If $i \leq n-11$, then $\alpha^2 = \varepsilon_{i+3} - \varepsilon_{n-7} \in \Gamma_{\beta_{(i+3)/2}}^0 \cap O_2$, $\theta(\alpha^2) = \varepsilon_{i+2} + \varepsilon_{n-7} \in O_2$, $\alpha^3 = \varepsilon_{n-6} - \varepsilon_{i+2} \in \Gamma_{\varepsilon_{n-6}-\varepsilon_{n-8}}^0 \cap O_2$ and by paragraph (a) above, $\alpha^3 \in O_{st}$. Hence by remark (4) of 3.6 one has that $\alpha \in O_{st}$.

If $\alpha = \varepsilon_i - \varepsilon_{n-6}$ for $1 \leq i \leq n-7$, one has that $\alpha \in O_{st}$ since $\theta(\alpha) \in O_{st}$ by the above.

Hence condition (4) of lemma 3.8 is satisfied for all roots in $\Gamma_{\varepsilon_{n-3}-\varepsilon_{n-6}}^0$.

(c) The roots in $\Gamma_{\varepsilon_{n-4}-\varepsilon_{n-5}}^0$.

• Let $\alpha = \varepsilon_{n-3} - \varepsilon_{n-5} \in \Gamma_{\varepsilon_{n-4}-\varepsilon_{n-5}}^0$. Then $\theta(\alpha) = \varepsilon_{n-4} - \varepsilon_{n-3}$ and one verifies that $\alpha \in O_{cyc}$.

Indeed let $\beta = \varepsilon_{n-4} - \varepsilon_{n-1} \in \Gamma_{\varepsilon_{n-4}+\varepsilon_{n-5}}^0$ and $\gamma = \varepsilon_{n-3} + \varepsilon_{n-5} \in \Gamma_{\varepsilon_{n-1}+\varepsilon_{n-3}}^0$. Then α, β, γ verify in this order the cyclic relations (i)-(iii) of 3.7 and $\beta, \theta(\beta), \gamma, \theta(\gamma)$ belong to O_2 . Moreover α and $\theta(\alpha)$ belong to O_3 (unless $n = 6$, in which case $\alpha \in O_2$). Assume that $n \geq 8$ and let $\tilde{\alpha} = \varepsilon_{n-5} - \varepsilon_{n-6} \in \Gamma_{\beta_{n/2-2}}^0 \cap O_2 \cap S_\alpha$. Then $\theta(\tilde{\alpha}) = \varepsilon_{n-4} + \varepsilon_{n-6} \in O_2$, $\tilde{\alpha}^1 = \varepsilon_{n-7} - \varepsilon_{n-4} \in \Gamma_{\varepsilon_{n-2}-\varepsilon_{n-4}}^0 \cap O_2$ and $\theta(\tilde{\alpha}^1) = \varepsilon_{n-2} - \varepsilon_{n-7} \in O_1$. Hence $\tilde{\alpha}$ satisfies the stationary condition. Similarly $\widetilde{\theta(\alpha)} = \varepsilon_n - \varepsilon_{n-4} \in \Gamma_{\varepsilon_{n-2}-\varepsilon_{n-4}}^0 \cap O_2 \cap S_{\theta(\alpha)}$ is such that $\theta(\widetilde{\theta(\alpha)}) = \varepsilon_{n-2} - \varepsilon_n \in O_1$. Hence $\widetilde{\theta(\alpha)}$ satisfies the stationary condition.

If now $\alpha = \varepsilon_{n-4} - \varepsilon_{n-3}$ then $\alpha \in O_{cyc}$ by remark (1) of 3.7 since $\theta(\alpha) \in O_{cyc}$ by the above.

• Let $\alpha = \varepsilon_{n-4} - \varepsilon_{2i} \in \Gamma_{\varepsilon_{n-4}-\varepsilon_{n-5}}^0$, $1 \leq i \leq n/2 - 3$. One verifies that $\alpha \in O_{cyc}$. Indeed let $\beta = \varepsilon_{2i-1} - \varepsilon_{n-5} \in \Gamma_{\beta_i}^0$ and $\gamma = \varepsilon_{2i-1} + \varepsilon_{n-5} \in \Gamma_{\varepsilon_{n-4}+\varepsilon_{n-5}}^0$. Then α, β, γ verify in this order the cyclic relations (i)-(iii) of 3.7 and $\beta, \theta(\beta), \gamma, \theta(\gamma)$ belong to O_2 . Moreover α and $\theta(\alpha)$ belong to O_3 (unless $2i = n - 6$, in which case $\alpha \in O_2$ or $i = 1$, in which case $\theta(\alpha) \in O_2$). Assume that $2i \leq n - 8$ and let $\tilde{\alpha} = \varepsilon_{2i+2} - \varepsilon_{n-4} \in \Gamma_{\varepsilon_{n-2}-\varepsilon_{n-4}}^0 \cap O_3 \cap S_\alpha$. Then $\theta(\tilde{\alpha}) = \varepsilon_{n-2} - \varepsilon_{2i+2} \in O_2$ and, if $2i \leq n - 10$, $\tilde{\alpha}^1 = \varepsilon_{2i+4} - \varepsilon_{n-2} \in \Gamma_{\beta_{i+2}}^0 \cap O_2$ and $\theta(\tilde{\alpha}^1) = \varepsilon_{2i+3} + \varepsilon_{n-2} \in O_1$ and if $2i = n - 8$, then $\tilde{\alpha}^1 = \varepsilon_{n-3} - \varepsilon_{n-2} \in \Gamma_{\varepsilon_{n-3}+\varepsilon_{n-1}}^0 \cap O_2$ and $\theta(\tilde{\alpha}^1) = \varepsilon_{n-2} + \varepsilon_{n-1} \in O_1$.

Let $\tilde{\alpha}' = \varepsilon_{2i+1} + \varepsilon_{n-4} \in \Gamma_{\beta_{n/2-2}}^0 \cap O_2 \cap S_{\tilde{\alpha}}$. Then $\theta(\tilde{\alpha}') = \varepsilon_{n-5} - \varepsilon_{2i+1} \in O_1$. Hence $\tilde{\alpha}$ satisfies the stability condition and then by the above $\tilde{\alpha}$ satisfies the stationary condition.

Similarly if $i \geq 2$, then $\widetilde{\theta(\alpha)} = \varepsilon_{n-5} - \varepsilon_{2i-2} \in \Gamma_{\beta_{n/2-2}}^0 \cap O_2 \cap S_{\theta(\alpha)}$, $\theta(\widetilde{\theta(\alpha)}) = \varepsilon_{n-4} + \varepsilon_{2i-2} \in O_2$, $\widetilde{\theta(\alpha)}^1 = \varepsilon_{2i-3} - \varepsilon_{n-4} \in \Gamma_{\varepsilon_{n-2}-\varepsilon_{n-4}}^0 \cap O_2$ and $\theta(\widetilde{\theta(\alpha)}^1) = \varepsilon_{n-2} - \varepsilon_{2i-3} \in O_1$. Hence $\widetilde{\theta(\alpha)}$ satisfies the stationary condition.

If now $\alpha = \varepsilon_{2i} - \varepsilon_{n-5}$ for $1 \leq i \leq n/2 - 3$, then $\alpha \in O_{cyc}$ since $\theta(\alpha) \in O_{cyc}$ by the above and remark (1) of 3.7.

Hence condition (4) of lemma 3.8 is satisfied for all roots in $\Gamma_{\varepsilon_{n-4}-\varepsilon_{n-5}}^0$.

(d) The roots in $\Gamma_{\varepsilon_{n-2}-\varepsilon_{n-4}}^0$.

• Let $\alpha = \varepsilon_{n-2} - \varepsilon_{n-1} \in \Gamma_{\varepsilon_{n-2}-\varepsilon_{n-4}}^0$. Then $\alpha \in O_1$. Moreover $\theta(\alpha) = \varepsilon_{n-1} - \varepsilon_{n-4} \in O_2$, $\alpha^1 = \varepsilon_{n-3} + \varepsilon_{n-4} \in \Gamma_{\beta_{n/2-2}}^0 \cap O_2$ and $\theta(\alpha^1) = \varepsilon_{n-5} - \varepsilon_{n-3} \in O_1$ (one has that $\varepsilon_n - \varepsilon_{n-5} \notin \Gamma$). Hence $\alpha \in O_{st}$.

If $\alpha = \varepsilon_{n-1} - \varepsilon_{n-4}$ then $\alpha \in O_{st}$ since $\theta(\alpha) \in O_{st}$ by the above.

• Let $\alpha = \varepsilon_{n-2} - \varepsilon_n \in \Gamma_{\varepsilon_{n-2}-\varepsilon_{n-4}}^0$.

Let $\beta = \varepsilon_{n-4} - \varepsilon_{n-3} \in \Gamma_{\varepsilon_{n-4}-\varepsilon_{n-5}}^0$. By paragraph (c) above one has that $\beta \in O_3 \cap O_{cyc}$. Moreover set $\theta(\alpha) = \tilde{\beta}$. One has that $\theta(\alpha) = \varepsilon_n - \varepsilon_{n-4} \in O_2 \cap S_\beta$ and $\alpha \in O_1$, hence $\theta(\alpha) = \tilde{\beta}$ satisfies the stationary condition.

Hence the last part of condition (4) of lemma 3.8 for α is satisfied.

If now $\alpha = \varepsilon_n - \varepsilon_{n-4}$, then by the above the last part of condition (4) of lemma 3.8 for α is satisfied since it is for $\theta(\alpha)$.

- Let $\alpha = \varepsilon_{n-2} - \varepsilon_i \in \Gamma_{\varepsilon_{n-2}-\varepsilon_{n-4}}^0$ for $1 \leq i \leq n-5$.

First suppose that $i = n-5$. Then one checks that $\theta(\alpha) = \varepsilon_{n-5} - \varepsilon_{n-4} \in O_1$ and that $\alpha \in O_2$ with $\alpha^{(1)} = \varepsilon_{n-4} - \varepsilon_{n-2} \in \Gamma_{\beta_{n/2-2}}^0 \cap O_2$ and $\theta(\alpha^{(1)}) = \varepsilon_{n-5} + \varepsilon_{n-2} \in O_1$. Hence $\alpha \in O_{st}$.

Now suppose that i is odd, $i \leq n-7$ and let $\beta = \varepsilon_{i+3} - \varepsilon_{n-5}$ if $i \leq n-9$, resp. $\beta = \varepsilon_{n-3} - \varepsilon_{n-5}$ if $i = n-7$. By paragraph (c) above one has that $\beta \in \Gamma_{\varepsilon_{n-4}-\varepsilon_{n-5}}^0 \cap O_3 \cap O_{cyc}$. Let $\tilde{\beta} = \varepsilon_{n-5} - \varepsilon_{i+1}$. Then $\tilde{\beta} \in \Gamma_{\beta_{n/2-2}}^0 \cap S_\beta \cap O_2$, $\theta(\tilde{\beta}) = \varepsilon_{n-4} + \varepsilon_{i+1} \in O_2$ and $\tilde{\beta}^1 = \varepsilon_i - \varepsilon_{n-4} \in \Gamma_{\varepsilon_{n-2}-\varepsilon_{n-4}}^0 \cap O_2$ and $\alpha = \theta(\tilde{\beta}^1) \in O_1$. Hence $\tilde{\beta}$ satisfies the stationary condition and $\theta(\alpha) = \tilde{\beta}^1$.

Hence α satisfies the last part of condition (4) of lemma 3.8.

Now suppose that i is even, $i \geq 4$, and let $\beta = \varepsilon_{n-4} - \varepsilon_{i-2}$. Then $\beta \in \Gamma_{\varepsilon_{n-4}-\varepsilon_{n-5}}^0 \cap O_3 \cap O_{cyc}$ by paragraph (c) above. Let $\tilde{\beta} = \varepsilon_i - \varepsilon_{n-4}$. Then $\tilde{\beta} = \theta(\alpha) \in S_\beta \cap O_3$ and by paragraph (c) above satisfies the stationary condition.

Hence α satisfies the last part of condition (4) of lemma 3.8.

Assume that $i = 2$. Then $\alpha \in O_1$, $\theta(\alpha) = \varepsilon_2 - \varepsilon_{n-4} \in O_2$, $\alpha^1 = \varepsilon_1 + \varepsilon_{n-4} \in \Gamma_{\beta_{n/2-2}}^0 \cap O_2$, $\theta(\alpha^1) = \varepsilon_{n-5} - \varepsilon_1 \in O_1$. Hence $\alpha \in O_{st}$.

If now $\alpha = \varepsilon_i - \varepsilon_{n-4}$, $1 \leq i \leq n-5$, then $\alpha \in O_{st}$ if $i = n-5$ or $i = 2$ since $\theta(\alpha) \in O_{st}$ by the above and if $1 \leq i \leq n-6$, $i \neq 2$, then α satisfies the last part of condition (4) of lemma 3.8 since $\theta(\alpha)$ does.

Hence condition (4) of lemma 3.8 is satisfied for all roots in $\Gamma_{\varepsilon_{n-2}-\varepsilon_{n-4}}^0$.

(e) The roots in $\Gamma_{\varepsilon_n-\varepsilon_{n-3}}^0$.

- Let $\alpha = \varepsilon_n - \varepsilon_{n-2}$. Then α and $\theta(\alpha) = \varepsilon_{n-2} - \varepsilon_{n-3}$ belong to O_1 . Hence α and $\theta(\alpha)$ are stationary roots.

- Let $\alpha = \varepsilon_n - \varepsilon_{n-1}$. Then α and $\theta(\alpha) = \varepsilon_{n-1} - \varepsilon_{n-3}$ belong to O_1 . Hence α and $\theta(\alpha)$ are stationary roots.

Hence, for $n = 6$, condition (3) of lemma 3.8 is satisfied (in this case, $\Gamma_{\varepsilon_6-\varepsilon_3}^0 = O^-$) and condition (4) in this case is empty.

- Let $\alpha = \varepsilon_n - \varepsilon_i$, $1 \leq i \leq n-6$.

Assume first that i is even. Then $\alpha \in O_2$ and $\theta(\alpha) \in O_3$ (unless $i = 2$, in which case $\theta(\alpha) \in O_2$, $\alpha^1 = \varepsilon_1 + \varepsilon_{n-3} \in \Gamma_{\varepsilon_{n-3}+\varepsilon_{n-1}}^0 \cap O_2$ and $\theta(\alpha^1) \in O_1$, hence α satisfies the stationary condition) and for $i \geq 4$, $\theta(\alpha)$ verifies the stability condition. Indeed

$\theta(\alpha)' = \varepsilon_{i-1} + \varepsilon_{n-3} \in \Gamma_{\varepsilon_{n-3} + \varepsilon_{n-1}}^0 \cap O_2 \cap S_{\theta(\alpha)}$ and $\theta(\theta(\alpha)') = \varepsilon_{n-1} - \varepsilon_{i-1} \in O_1$. Moreover if $i = n - 6$, then $\alpha^{(1)} = \varepsilon_{n-3} - \varepsilon_n \in \Gamma_{\varepsilon_{n-3} + \varepsilon_{n-1}}^0 \cap O_2$ and $\theta(\alpha^{(1)}) = \varepsilon_n + \varepsilon_{n-1} \in O_1$, and if $i \leq n - 8$, then $\alpha^{(1)} = \varepsilon_{i+2} - \varepsilon_n \in \Gamma_{\beta(i+2)/2}^0 \cap O_2$ and $\theta(\alpha^{(1)}) = \varepsilon_{i+1} + \varepsilon_n \in O_1$. Hence $\theta(\alpha)$ satisfies the stationary condition. On the other hand if $i \geq 4$ then $\alpha^1 = \varepsilon_{n-3} - \varepsilon_{i-2} \in \Gamma_{\varepsilon_{n-3} - \varepsilon_{n-6}}^0 \cap O_3 \cap O_{st}$ by paragraph (b) above. By remark (4) of 3.6 or directly for $i = 2$, one deduces that $\alpha \in O_{st}$.

Assume now that i is odd. Then $\alpha \in O_1$ and $\theta(\alpha) \in O_2$ (which implies that the root $\theta(\alpha)$ satisfies the stationary condition). Moreover $\alpha^1 = \varepsilon_{i+1} + \varepsilon_{n-3} \in \Gamma_{\varepsilon_{n-3} + \varepsilon_{n-1}}^0 \cap O_2$, $\theta(\alpha^1) = \varepsilon_{n-1} - \varepsilon_{i+1}$. If $i = n - 7$ then $\theta(\alpha^1) \in O_1$, hence α satisfies the stationary condition. If $i \leq n - 9$ then $\theta(\alpha^1) \in O_2$ and $\alpha^2 = \varepsilon_{i+3} - \varepsilon_{n-1} \in \Gamma_{\beta(i+3)/2}^0 \cap O_2$, $\theta(\alpha^2) = \varepsilon_{i+2} + \varepsilon_{n-1} \in O_2$ and $\alpha^3 = \varepsilon_{n-3} - \varepsilon_{i+2} \in \Gamma_{\varepsilon_{n-3} - \varepsilon_{n-6}}^0 \cap O_{st}$ by paragraph (b) above. By remark (4) of 3.6 or directly for $i = n - 7$, one deduces that $\alpha \in O_{st}$.

If now $\alpha = \varepsilon_i - \varepsilon_{n-3}$, $1 \leq i \leq n - 6$, then by the above one has that $\alpha \in O_{st}$ since $\theta(\alpha) \in O_{st}$.

Hence condition (4) of lemma 3.8 is satisfied for all roots in $\Gamma_{\varepsilon_n - \varepsilon_{n-3}}^0$, for $n \geq 8$.

(f) The roots in $\Gamma_{\varepsilon_{n-3} + \varepsilon_{n-1}}^0$.

- Let $\alpha = \varepsilon_{n-3} + \varepsilon_n$. Then one checks that α and $\theta(\alpha) = -\varepsilon_n + \varepsilon_{n-1}$ lie in O_{st} since both belong to O_1 .

- Let $\alpha = \varepsilon_{n-3} - \varepsilon_n$. Then $\theta(\alpha) = \varepsilon_n + \varepsilon_{n-1} \in O_1$, $\alpha \in O_2$ (unless $n = 6$, in which case $\alpha \in O_1$, hence $\theta(\alpha)$ satisfies the stationary condition). Then the root α satisfies the stationary condition. Moreover, if $n \geq 8$, $\alpha^{(1)} = \varepsilon_n - \varepsilon_{n-6} \in \Gamma_{\varepsilon_n - \varepsilon_{n-3}}^0 \cap O_2 \cap O_{st}$ by paragraph (e) above. Then by remark (4) of 3.6 or directly for $n = 6$, one has that $\alpha \in O_{st}$ and then $\theta(\alpha) \in O_{st}$.

- Let $\alpha = \varepsilon_{n-3} - \varepsilon_{n-2}$. Assume that $n \geq 10$ and let $\beta = \varepsilon_{n-4} - \varepsilon_{n-8} \in \Gamma_{\varepsilon_{n-4} - \varepsilon_{n-5}}^0 \cap O_3 \cap O_{cyc}$ by paragraph (c) above. One has that $\tilde{\beta} = \varepsilon_{n-6} - \varepsilon_{n-4} \in S_\beta$ and satisfies the stationary condition by paragraph (c) and moreover $\tilde{\beta}^1 = \alpha$. Hence α satisfies the last part of condition (4) of lemma 3.8 and then so does $\theta(\alpha) = \varepsilon_{n-2} + \varepsilon_{n-1}$.

Assume that $n = 6$. Then $\alpha = \varepsilon_3 - \varepsilon_4 \in O_1$ and $\theta(\alpha) = \varepsilon_4 + \varepsilon_5 \in O_1$ then $\alpha, \theta(\alpha) \in O_{st}$.

Assume that $n = 8$. Then $\alpha = \varepsilon_5 - \varepsilon_6 \in O_2$ and $\theta(\alpha) = \varepsilon_6 + \varepsilon_7 \in O_1$. Hence α satisfies the stationary condition. Moreover $\alpha^{(1)} = \varepsilon_6 - \varepsilon_2 \in \Gamma_{\varepsilon_6 - \varepsilon_4}^0 \cap O_2$, $\theta(\alpha^{(1)}) = \varepsilon_2 - \varepsilon_4 \in O_2$, $\alpha^{(2)} = \varepsilon_1 + \varepsilon_4 \in \Gamma_{\varepsilon_3 + \varepsilon_4}^0 \cap O_2$, $\theta(\alpha^{(2)}) = \varepsilon_3 - \varepsilon_1 \in O_1$. Hence $\theta(\alpha)$ satisfies the stationary condition and then $\alpha \in O_{st}$ (and of course $\theta(\alpha) \in O_{st}$).

- Let $\alpha = \varepsilon_{n-3} + \varepsilon_{n-2}$. Then α and $\theta(\alpha) = \varepsilon_{n-1} - \varepsilon_{n-2}$ lie in O_{st} since both lie in O_1 .

- Let $\alpha = \varepsilon_{n-1} - \varepsilon_i$, $1 \leq i \leq n - 5$.

Assume first that $i = n - 6$. Then one checks that $\alpha \in O_1$ since $\varepsilon_{n-3} - \varepsilon_{n-1} \notin \Gamma$ and $\theta(\alpha) = \varepsilon_{n-3} + \varepsilon_{n-6} \in O_2$, hence $\theta(\alpha)$ satisfies the stationary condition. Moreover

$\alpha^1 = \varepsilon_{n-7} - \varepsilon_{n-3} \in \Gamma_{\varepsilon_n - \varepsilon_{n-3}}^0 \cap O_{st}$ by paragraph (e) above. Then by remark (4) of 3.6, $\alpha \in O_{st}$.

Assume that i is even, $2 \leq i \leq n-8$. Then $\alpha \in O_2$, $\alpha^{(1)} = \varepsilon_{i+2} - \varepsilon_{n-1} \in \Gamma_{\beta_{(i+2)/2}}^0 \cap O_2$, $\theta(\alpha^{(1)}) = \varepsilon_{i+1} + \varepsilon_{n-1} \in O_2$, $\alpha^{(2)} = \varepsilon_{n-3} - \varepsilon_{i+1} \in \Gamma_{\varepsilon_{n-3} - \varepsilon_{n-6}}^0 \cap O_{st}$ by paragraph (b) above.

On the other hand $\theta(\alpha) = \varepsilon_i + \varepsilon_{n-3} \in O_2$, $\alpha^1 = \varepsilon_{i-1} - \varepsilon_{n-3} \in \Gamma_{\varepsilon_n - \varepsilon_{n-3}}^0 \cap O_{st}$ by paragraph (e) above. Hence by remark (5) of 3.6, $\alpha \in O_{st}$.

Assume that $i = n-5$. Then $\alpha \in O_2$ and $\theta(\alpha) = \varepsilon_{n-5} + \varepsilon_{n-3} \in O_{cyc}$ by paragraph (c) above and remark (1) of 3.7. Again remark (1) of 3.7 implies that $\alpha \in O_{cyc}$.

Assume that $1 \leq i \leq n-7$ and i odd. Then $\alpha \in O_1$ and $\theta(\alpha) = \varepsilon_i + \varepsilon_{n-3} \in O_2$, hence $\theta(\alpha)$ satisfies the stationary condition. Moreover $\alpha^1 = \varepsilon_{i+1} - \varepsilon_{n-3} \in \Gamma_{\varepsilon_n - \varepsilon_{n-3}}^0 \cap O_{st}$ by paragraph (e) above. Hence by remark (4) of 3.6, $\alpha \in O_{st}$.

If now $\alpha = \varepsilon_i + \varepsilon_{n-3}$, $1 \leq i \leq n-5$. By the above, if $i \neq n-5$, $\theta(\alpha) \in O_{st}$. Then $\alpha \in O_{st}$. If $i = n-5$, then by the above, $\alpha \in O_{cyc}$.

Hence condition (4) of lemma 3.8 is satisfied for all roots in $\Gamma_{\varepsilon_{n-3} + \varepsilon_{n-1}}^0$.

(g) The roots in $\Gamma_{\beta_{n/2-2}}^0$.

One has that $\beta_{n/2-2} = \varepsilon_{n-5} + \varepsilon_{n-4}$ and $\Gamma_{\beta_{n/2-2}}^0 = \{\varepsilon_{n-5} + \varepsilon_i, \varepsilon_{n-4} - \varepsilon_i, \varepsilon_{n-5} - \varepsilon_j, \varepsilon_j + \varepsilon_{n-4}, \varepsilon_{n-5} + \varepsilon_{2k-1}, \varepsilon_{n-4} - \varepsilon_{2k-1} \mid n-2 \leq i \leq n, 1 \leq j \leq n, j \notin \{n-5, n-4\}, 1 \leq k \leq n/2-3\}$.

• Let $\alpha = \varepsilon_{n-5} + \varepsilon_i$, with $n-2 \leq i \leq n$.

Assume first that $i = n-2$. Then $\alpha \in O_1$ and $\theta(\alpha) = \varepsilon_{n-4} - \varepsilon_{n-2} \in O_2$, hence $\theta(\alpha)$ satisfies the stationary condition. Moreover $\alpha^1 = \varepsilon_{n-2} - \varepsilon_{n-5} \in \Gamma_{\varepsilon_{n-2} - \varepsilon_{n-4}}^0 \cap O_{st}$ by paragraph (d) above. Hence remark (4) of 3.6 implies that $\alpha \in O_{st}$. If $i = n$ then both α and $\theta(\alpha)$ belong to O_1 (since $\varepsilon_n - \varepsilon_{n-5} \notin \Gamma$). Hence $\alpha \in O_{st}$.

Now assume that $i = n-1$, then by paragraph (c) above one has that $\theta(\alpha) = \varepsilon_{n-4} - \varepsilon_{n-1} \in O_{cyc}$ and then by remark (1) of 3.7 one has that $\alpha \in O_{cyc}$.

Now if $\alpha = \varepsilon_{n-4} - \varepsilon_i$, with $n-2 \leq i \leq n$.

Then by the above if $i \in \{n-2, n\}$, then $\theta(\alpha) \in O_{st}$, hence $\alpha \in O_{st}$, and if $i = n-1$, then $\alpha \in O_{cyc}$.

• Let $\alpha = \varepsilon_{n-5} - \varepsilon_j$, with $1 \leq j \leq n$, $j \neq n-5$, $j \neq n-4$.

Assume first that $j = n-3$. Then $\alpha \in O_1$ and $\theta(\alpha) \in O_2$, which implies that $\theta(\alpha)$ satisfies the stationary condition. Moreover $\alpha^1 = \varepsilon_{n-1} - \varepsilon_{n-4} \in \Gamma_{\varepsilon_{n-2} - \varepsilon_{n-4}}^0 \cap O_{st}$ by paragraph (d) above. Hence by remark (4) of 3.6, $\alpha \in O_{st}$ and then $\theta(\alpha) = \varepsilon_{n-3} + \varepsilon_{n-4} \in O_{st}$.

Assume that $j \in \{n-2, n-1, n\}$. Then α and $\theta(\alpha) = \varepsilon_j + \varepsilon_{n-4}$ both belong to O_1 , which implies that α and $\theta(\alpha)$ belong to O_{st} (observe that $\varepsilon_{n-3} - \varepsilon_{n-4} \notin \Gamma$).

Assume now that $1 \leq j \leq n-6$, which implies that $n \geq 8$.

First assume that j is even, and let $\beta = \varepsilon_{j+2} - \varepsilon_{n-5}$, if $j \leq n-8$, resp. $\beta = \varepsilon_{n-3} - \varepsilon_{n-5}$, if $j = n-6$. Then $\beta \in \Gamma_{\varepsilon_{n-4}-\varepsilon_{n-5}}^0 \cap O_{cyc} \cap O_3$ by paragraph (c) above and $\tilde{\beta} = \alpha$ where $\tilde{\beta} \in S_\beta$ satisfies the stationary condition. Thus α (and then $\theta(\alpha)$) satisfies the last part of condition (4) of lemma 3.8.

Assume that $j = 1$. Then $\alpha = \varepsilon_{n-5} - \varepsilon_1 \in O_1$ and $\theta(\alpha) = \varepsilon_1 + \varepsilon_{n-4} \in O_2$, hence $\theta(\alpha)$ satisfies the stationary condition. Moreover $\alpha^1 = \varepsilon_2 - \varepsilon_{n-4} \in \Gamma_{\varepsilon_{n-2}-\varepsilon_{n-4}}^0 \cap O_2$, $\theta(\alpha^1) = \varepsilon_{n-2} - \varepsilon_2 \in O_1$. Thus α also satisfies the stationary condition and $\alpha, \theta(\alpha) \in O_{st}$.

Assume that $j \geq 3$, j odd. Then $\alpha \in O_1$ and $\theta(\alpha) = \varepsilon_j + \varepsilon_{n-4} \in O_2$. Moreover $\alpha^1 = \varepsilon_{j+1} - \varepsilon_{n-4} \in \Gamma_{\varepsilon_{n-2}-\varepsilon_{n-4}}^0 \cap O_3$. Let $\beta = \varepsilon_{n-4} - \varepsilon_{j-1} \in \Gamma_{\varepsilon_{n-4}-\varepsilon_{n-5}}^0 \cap O_{cyc} \cap O_3$ by paragraph (c) above, which also gives that $\tilde{\beta} = \alpha^1$ (where $\tilde{\beta} \in S_\beta$ satisfies the stationary condition). With the notations of 3.4, one has that $(\alpha^1)' = (\tilde{\beta})' = \theta(\alpha)$. Then $\theta(\alpha)$ and also α are stationary roots by remark (6) of 3.6.

• Let $\alpha = \varepsilon_{n-5} + \varepsilon_{2k-1}$, with $1 \leq k \leq n/2-3$. By paragraph (c) above and remark (1) of 3.7, one has that α (and then $\theta(\alpha)$) are cyclic roots.

Thus condition (4) of lemma 3.8 holds for all roots in $\Gamma_{\beta_{n/2-2}}^0$. One may also verify that condition (2) is satisfied in case $n = 6$, since one checks by the above that, for all $\alpha \in \Gamma_{\beta_1}^0 = \Gamma_{\beta_{n/2-2}}^0 = O^+$, $S_\alpha \cap O^+ = \{\theta(\alpha)\}$.

(h) The roots in $\Gamma_{\beta_i}^0$, with $1 \leq i \leq n/2-3$.

Observe that this implies that $n \geq 8$.

Recall that $\beta_i = \varepsilon_{2i-1} + \varepsilon_{2i}$ and observe that $\Gamma_{\beta_i}^0 = \{\varepsilon_{2i-1} + \varepsilon_{2j-1}, \varepsilon_{2i} - \varepsilon_{2j-1}, \varepsilon_{2i-1} - \varepsilon_{2k-1}, \varepsilon_{2i} + \varepsilon_{2k-1}, \varepsilon_{2i-1} \pm \varepsilon_u, \varepsilon_{2i} \mp \varepsilon_u \mid i+1 \leq j \leq n/2-3, i+1 \leq k \leq n/2-2, n-2 \leq u \leq n\} \sqcup \{\varepsilon_{2i-1} - \varepsilon_v, \varepsilon_{2i} + \varepsilon_v \mid 1 \leq v \leq 2i-2\}$.

• Let $\alpha = \varepsilon_{2i-1} - \varepsilon_v$, with $1 \leq v \leq 2i-2$ (which implies that $i \geq 2$).

First assume that v is even. Then $\theta(\alpha) = \varepsilon_v + \varepsilon_{2i} \in O_2$ and $\alpha^1 = \varepsilon_{v-1} - \varepsilon_{2i}$, which belongs to $\Gamma_{\varepsilon_{2i+2}-\varepsilon_{2i}}^0 \cap O_2 \cap O_{st}$ by paragraph (a) above if $2i \leq n-8$, resp. to $\Gamma_{\varepsilon_{n-3}-\varepsilon_{n-6}}^0 \cap O_2 \cap O_{st}$ if $2i = n-6$ by paragraph (b) above.

Moreover if $v = 2i-2$ then $\alpha \in O_1$ and $\theta(\alpha)$ satisfies the stationary condition, and if $v \leq 2i-4$, then $\alpha \in O_2$ and $\alpha^{(1)} = \varepsilon_{v+2} - \varepsilon_{2i-1} \in \Gamma_{\beta_{v/2+1}}^0 \cap O_2$, $\theta(\alpha^{(1)}) = \varepsilon_{2i-1} + \varepsilon_{v+1} \in O_2$ and $\alpha^{(2)} = \varepsilon_{2i} - \varepsilon_{v+1} \in \Gamma_{\varepsilon_{2i}-\varepsilon_{2i-2}}^0 \cap O_2 \cap O_{st}$ by paragraph (a) above.

Hence by remark (4) or (5) of 3.6 one deduces that α (and also $\theta(\alpha)$) belongs to O_{st} .

Now assume that v is odd. Then $\alpha \in O_1$ and $\theta(\alpha) \in O_2$ which implies that $\theta(\alpha)$ satisfies the stationary condition. Moreover $\alpha^1 = \varepsilon_{v+1} - \varepsilon_{2i}$, which belongs to $\Gamma_{\varepsilon_{2i+2}-\varepsilon_{2i}}^0 \cap O_3 \cap O_{st}$ by paragraph (a) above if $2i \leq n-8$, resp. to $\Gamma_{\varepsilon_{n-3}-\varepsilon_{n-6}}^0 \cap O_3 \cap O_{st}$ by paragraph (b) above, if $2i = n-6$. By remark (4) of 3.6 one deduces that α (and then $\theta(\alpha)$) belongs to O_{st} .

• Let $\alpha = \varepsilon_{2i-1} + \varepsilon_{2j-1}$, with $i+1 \leq j \leq n/2-3$. Then α and $\theta(\alpha) = \varepsilon_{2i} - \varepsilon_{2j-1}$ belong to O_2 (unless $i=1$, in which case $\theta(\alpha) \in O_1$ and α satisfies the stationary condition) and one has that $\alpha^{(1)} = \varepsilon_{2j} - \varepsilon_{2i-1} \in \Gamma_{\varepsilon_{2j}-\varepsilon_{2i-1}}^0 \cap O_{st}$ by paragraph (a) above and if $i \geq 2$, $\alpha^1 = \varepsilon_{2j-1} - \varepsilon_{2i-2} \in \Gamma_{\beta_j}^0 \cap O_{st}$ by the above. Remark (4) or (5) of 3.6 implies that α (and also $\theta(\alpha)$) belongs to O_{st} .

• Let $\alpha = \varepsilon_{2i-1} - \varepsilon_{2k-1}$, with $i+1 \leq k \leq n/2-2$.

First assume that $k = n/2-2$. Then by paragraph (c) above, one has that $\alpha = \varepsilon_{2i-1} - \varepsilon_{n-5} \in O_{cyc} \cap O_2$ and by remark (1) of 3.7 one has also that $\theta(\alpha) \in O_{cyc}$.

Now assume that $k \leq n/2-3$. Then $\alpha \in O_1$ and $\theta(\alpha) \in O_2$ (unless $k = i+1$, in which case $\theta(\alpha) = \varepsilon_{2i} + \varepsilon_{2k-1} \in O_1$ and then α satisfies the stationary condition). Thus $\theta(\alpha)$ satisfies the stationary condition. Moreover if $k \geq i+2$, then $\alpha^1 = \varepsilon_{2k} - \varepsilon_{2i} \in \Gamma_{\varepsilon_{2k}-\varepsilon_{2i-2}}^0 \cap O_{st}$ by paragraph (a) above. Hence by remark (4) of 3.6 or directly one deduces that α (and also $\theta(\alpha)$) belongs to O_{st} .

• Let $\alpha = \varepsilon_{2i-1} + \varepsilon_u$, with $n-2 \leq u \leq n$.

First assume that $u = n-2$ and that $i = 1$. Then $\alpha = \varepsilon_1 + \varepsilon_{n-2} \in O_1$ and $\theta(\alpha) = \varepsilon_2 - \varepsilon_{n-2} \in O_1$ and $\alpha, \theta(\alpha) \in O_{st}$.

Assume that $u = n-2$ and $i = 2$. Then $\alpha = \varepsilon_3 + \varepsilon_{n-2} \in O_1$ and $\theta(\alpha) = \varepsilon_4 - \varepsilon_{n-2} \in O_2$, hence $\theta(\alpha)$ satisfies the stationary condition. Moreover $\alpha^1 = \varepsilon_{n-2} - \varepsilon_2 \in \Gamma_{\varepsilon_{n-2}-\varepsilon_{n-4}}^0 \cap O_{st}$ if $n \geq 10$, resp. $\alpha^1 = \varepsilon_6 - \varepsilon_3 \in \Gamma_{\varepsilon_6-\varepsilon_4}^0 \cap O_{st}$ if $n = 8$, by paragraph (d) above. Hence $\alpha, \theta(\alpha) \in O_{st}$ by remark (4) of 3.6.

Assume that $u = n-2$ and $i \geq 3$. Let $\beta = \varepsilon_{n-4} - \varepsilon_{2i-4} \in \Gamma_{\varepsilon_{n-4}-\varepsilon_{n-5}}^0 \cap O_3 \cap O_{cyc}$ by paragraph (c) above. One has that $\tilde{\beta} = \varepsilon_{2i-2} - \varepsilon_{n-4} \in S_\beta$ satisfies the stationary condition and $\theta(\alpha) = \tilde{\beta}^1$.

Assume that $u = n-1$. Then $\alpha \in O_2$ and $\alpha^{(1)} = \varepsilon_{n-3} - \varepsilon_{2i-1} \in \Gamma_{\varepsilon_{n-3}-\varepsilon_{n-6}}^0 \cap O_{st}$ by paragraph (b) above. If $i = 1$ then $\theta(\alpha) \in O_1$ and α satisfies the stationary condition, and if $i \geq 2$, then $\theta(\alpha) \in O_2$ with $\alpha^1 = \varepsilon_{n-1} - \varepsilon_{2i-2} \in \Gamma_{\varepsilon_{n-3}+\varepsilon_{n-1}}^0 \cap O_{st}$ by paragraph (f) above. By remark (4) or (5) of 3.6 one has that $\alpha, \theta(\alpha) \in O_{st}$.

Assume that $u = n$. Then $\alpha \in O_1$ and if $i = 1$, $\theta(\alpha) \in O_1$. If $i \geq 2$, then $\theta(\alpha) \in O_2$ and $\alpha^1 = \varepsilon_n - \varepsilon_{2i-2} \in \Gamma_{\varepsilon_n-\varepsilon_{n-3}}^0 \cap O_{st}$ by paragraph (e) above. Hence $\alpha, \theta(\alpha) \in O_{st}$ by remark (4) of 3.6.

• Let $\alpha = \varepsilon_{2i-1} - \varepsilon_u$, with $n-2 \leq u \leq n$.

If $u \in \{n-2, n\}$ then $\alpha, \theta(\alpha) \in O_1$ and $\alpha, \theta(\alpha) \in O_{st}$.

Assume that $u = n-1$. Then $\alpha \in O_1$ and $\theta(\alpha) = \varepsilon_{n-1} + \varepsilon_{2i} \in O_2$ (unless $i = n/2-3$, in which case $\theta(\alpha) \in O_1$) and if $i \leq n/2-4$ then $\alpha^1 = \varepsilon_{n-3} - \varepsilon_{2i} \in \Gamma_{\varepsilon_{n-3}-\varepsilon_{n-6}}^0 \cap O_{st}$ by paragraph (b) above. Hence $\alpha, \theta(\alpha) \in O_{st}$.

Thus condition (4) of lemma 3.8 holds for all roots in $\Gamma_{\beta_i}^0$, with $1 \leq i \leq n/2-3$. Also condition (2) for $i = 1$ and $n \geq 8$ holds since by the above one may verify that, if $\alpha \in \Gamma_{\beta_1}^0$, then $S_\alpha \cap \Gamma_{\beta_1}^0 = \{\theta(\alpha)\}$.

6.4. The set T . Recall that we denote by T the complement of the set $\Gamma = \bigsqcup_{\gamma \in S} \Gamma_\gamma$ in $\Delta^+ \sqcup \Delta_\pi^-$. One checks that $T = \{\varepsilon_{n-3} - \varepsilon_{n-1}, \varepsilon_{n-2} + \varepsilon_n, \varepsilon_n - \varepsilon_{n-5}, \varepsilon_{n-3} - \varepsilon_{n-4}, \varepsilon_{n-2k} - \varepsilon_{n-2k-1} \mid 3 \leq k \leq n/2 - 1\}$. Then $|T| = n/2 + 1$.

Moreover the $\langle \mathbf{ij} \rangle$ -orbits in π are $\Gamma_t = \{\alpha_t, \alpha_{n-t}\}$ for all $1 \leq t \leq n/2 - 1$, $\Gamma_{n/2} = \{\alpha_{n/2}\}$ and $\Gamma_n = \{\alpha_n\}$. They are $n/2 + 1$ in number, hence $|T| = \text{ind } \mathfrak{p}$.

Remark.

All conditions of lemma 3.1 are satisfied. Hence defining $h \in \mathfrak{h}_\Lambda$ by $\gamma(h) = -1$ for all $\gamma \in S$, and setting $y = \sum_{\gamma \in S} x_\gamma$ we obtain an adapted pair (h, y) for $\mathfrak{p}_{\pi', \Lambda}^-$.

6.5. The eigenvalues of $\text{ad } h$ on \mathfrak{g}_T .

We give an expansion of the semisimple element h :

For $n = 6$, one has

$h = -\varepsilon_2 + 5\varepsilon_3 - 2\varepsilon_4 - 6\varepsilon_5 + 4\varepsilon_6$ and for $n \geq 8$, one has :

$$h = -n\varepsilon_1 + \sum_{k=1}^{n/2-4} (k-n)\varepsilon_{2k+1} + \sum_{k=1}^{n/2-3} (n-k)\varepsilon_{2k} - \varepsilon_{n-4} + (n/2+2)\varepsilon_{n-3} - 2\varepsilon_{n-2} - (n/2+3)\varepsilon_{n-1} + (n/2+1)\varepsilon_n.$$

Then the eigenvalues of $\text{ad } h$ on \mathfrak{g}_T are :

- $2(n-i) + 1 = (\varepsilon_{2i} - \varepsilon_{2i-1})(h)$ for all $1 \leq i \leq n/2 - 3$.
- $n + 5 = (\varepsilon_{n-3} - \varepsilon_{n-1})(h)$.
- $n/2 - 1 = (\varepsilon_{n-2} + \varepsilon_n)(h)$.
- $n/2 + 1 = (\varepsilon_n - \varepsilon_{n-5})(h)$.
- $n/2 + 3 = (\varepsilon_{n-3} - \varepsilon_{n-4})(h)$.

From the first two equalities, we have that $n + 4 + 2k - 1$ is an eigenvalue of $\text{ad } h$ on \mathfrak{g}_T , for all $k \in \mathbb{N}$, with $1 \leq k \leq n/2 - 2$.

6.6. Polynomiality of $Y(\mathfrak{p})$.

6.6.1. The lower bound for $\text{ch } Y(\mathfrak{p})$. We will show that the lower bound for $Y(\mathfrak{p})$ is equal to

$$\prod_{\Gamma \in E(\pi')} (1 - e^{\delta_\Gamma})^{-1} = (1 - e^{-2\varpi_n})^{-3} (1 - e^{-4\varpi_n})^{-(n/2-2)} \quad (4)$$

Indeed, one checks that, for all t , with $1 \leq t \leq n-2$, $\varpi_t - \varpi'_t = (2t/n)\varpi_n$ and that $\varpi_{n-1} - \varpi'_{n-1} = ((n-2)/n)\varpi_n$. Then with the notations of Section 6.4 one has that, for all t , with $2 \leq t \leq n/2 - 1$, $\delta_{\Gamma_t} = -4\varpi_n$, whereas $\delta_{\Gamma_1} = \delta_{\Gamma_{n/2}} = \delta_{\Gamma_n} = -2\varpi_n$. Hence equality (4) holds.

6.6.2. The improved upper bound for $\text{ch } Y(\mathfrak{p})$. We now compute the improved upper bound for $\text{ch } Y(\mathfrak{p})$. We will show that it equals the lower bound, given by the right hand side of (4).

Recall that, if for every $\gamma \in T$, $t(\gamma)$ denotes the unique element in $\mathbb{Q}S$ such that $\gamma + t(\gamma)$ is a multiple of ϖ_n , then the improved upper bound is given by the product $\prod_{\gamma \in T} (1 - e^{-(\gamma + t(\gamma))})^{-1}$.

One verifies that :

- $t(\varepsilon_{n-2} + \varepsilon_n) = \sum_{1 \leq i \leq n/2-2} (\varepsilon_{2i-1} + \varepsilon_{2i}) + (\varepsilon_{n-3} + \varepsilon_{n-1})$ and that $\varepsilon_{n-2} + \varepsilon_n + t(\varepsilon_{n-2} + \varepsilon_n) = 2\varpi_n$.
- $t(\varepsilon_n - \varepsilon_{n-5}) = \sum_{1 \leq i \leq n/2-3} (\varepsilon_{2i-1} + \varepsilon_{2i}) + 2(\varepsilon_{n-5} + \varepsilon_{n-4}) + (\varepsilon_{n-3} + \varepsilon_{n-1}) + (\varepsilon_{n-2} - \varepsilon_{n-4})$ and $\varepsilon_n - \varepsilon_{n-5} + t(\varepsilon_n - \varepsilon_{n-5}) = 2\varpi_n$.
- $t(\varepsilon_{n-3} - \varepsilon_{n-4}) = \sum_{1 \leq i \leq n/2-3} (\varepsilon_{2i-1} + \varepsilon_{2i}) + 2(\varepsilon_{n-5} + \varepsilon_{n-4}) + (\varepsilon_{n-3} + \varepsilon_{n-1}) + (\varepsilon_n - \varepsilon_{n-3}) + (\varepsilon_{n-2} - \varepsilon_{n-4}) + (\varepsilon_{n-4} - \varepsilon_{n-5})$, and $\varepsilon_{n-3} - \varepsilon_{n-4} + t(\varepsilon_{n-3} - \varepsilon_{n-4}) = 2\varpi_n$.
- $t(\varepsilon_{n-3} - \varepsilon_{n-1}) = 2 \sum_{1 \leq i \leq n/2-3} (\varepsilon_{2i-1} + \varepsilon_{2i}) + 3(\varepsilon_{n-5} + \varepsilon_{n-4}) + 3(\varepsilon_{n-3} + \varepsilon_{n-1}) + 2(\varepsilon_n - \varepsilon_{n-3}) + 2(\varepsilon_{n-2} - \varepsilon_{n-4}) + (\varepsilon_{n-4} - \varepsilon_{n-5})$ and $\varepsilon_{n-3} - \varepsilon_{n-1} + t(\varepsilon_{n-3} - \varepsilon_{n-1}) = 4\varpi_n$.
- For $3 \leq k \leq n/2 - 1$, $t(\varepsilon_{n-2k} - \varepsilon_{n-2k-1}) = 2 \sum_{1 \leq i \leq n/2-3, i \neq n/2-k} (\varepsilon_{2i-1} + \varepsilon_{2i}) + 3(\varepsilon_{n-5} + \varepsilon_{n-4}) + 3(\varepsilon_{n-2k-1} + \varepsilon_{n-2k}) + 2(\varepsilon_{n-3} + \varepsilon_{n-1}) + 2(\varepsilon_n - \varepsilon_{n-3}) + 2(\varepsilon_{n-2} - \varepsilon_{n-4}) + (\varepsilon_{n-4} - \varepsilon_{n-5}) + 2(\varepsilon_{n-3} - \varepsilon_{n-6}) + 2 \sum_{3 \leq j \leq k-1} (\varepsilon_{n-2j} - \varepsilon_{n-2j-2})$ and $\varepsilon_{n-2k} - \varepsilon_{n-2k-1} + t(\varepsilon_{n-2k} - \varepsilon_{n-2k-1}) = 4\varpi_n$.

Thus the improved upper bound is equal to the right hand side of (4).

6.6.3. *Conclusion.* Recall 2.9 and 2.10. One can now give the following

Theorem. Let \mathfrak{g} be a simple Lie algebra of type D_n with n an even integer, $n \geq 6$, and let $\mathfrak{p} = \mathfrak{p}_{\pi', \Lambda}^-$ be the truncated maximal parabolic subalgebra of \mathfrak{g} associated to $\pi' = \pi \setminus \{\alpha_n\}$.

There exists an adapted pair (h, y) for \mathfrak{p} and an affine slice $y + \mathfrak{g}_T$ in \mathfrak{p}^* such that restriction of functions gives an isomorphism of algebras between $Y(\mathfrak{p})$ and the ring $R[y + \mathfrak{g}_T]$ of polynomial functions on $y + \mathfrak{g}_T$.

In particular $Y(\mathfrak{p})$ is a polynomial algebra over k , the degrees of a set of homogeneous generators are the eigenvalues plus one of $\text{ad } h$ on \mathfrak{g}_T (6.5) and the field $C(\mathfrak{p}_{\pi'}^-)$ of invariant fractions is a purely transcendental extension of k .

6.7. Let \mathfrak{g} be of type E_7 and let \mathfrak{p} be the truncated maximal parabolic subalgebra corresponding to $\pi' = \pi \setminus \{\alpha_3\}$. Let β_1 be the unique highest root of \mathfrak{g} and let $H_{\beta_1} = \{\beta \in \Delta^+ \mid (\beta, \beta_1) > 0\}$ be the maximal Heisenberg set of centre β_1 in Δ^+ . Then notice that the set $\Delta \setminus (H_{\beta_1} \sqcup -H_{\beta_1})$ is a root system of type D_6 and removing α_3 corresponds to removing the extremal root from a system of type D_6 .

Write $(a_1, a_2, a_3, a_4, a_5, a_6, a_7)$ for the root $\sum_{i=1}^7 a_i \alpha_i$ (with a_i some integers).

The sets S and T given in Sections 6.1 and 6.4 for type D_6 with $s = 6$ lead us to taking for S the set

$$S = \{\beta_1, (0, 1, 1, 2, 2, 2, 1), (0, 1, 1, 1, 1, 0, 0), \\ (0, -1, 0, -1, -1, 0, 0), (0, 0, 0, 0, -1, -1, 0), (0, 0, 0, 0, 0, 0, -1)\}$$

and for T the set

$$T = \{(-1, 0, 0, 0, 0, 0, 0), (0, 0, 0, 1, 1, 0, 0), (0, 0, 1, 1, 0, 0, 0), \\ (0, -1, 0, -1, -1, -1, -1), (0, 0, 0, 0, 0, -1, 0)\}.$$

More explicitly, we have added to the set S in type D_6 with $s = 6$ (rewritten with respect to the roots in type E_7) the highest root β_1 , and to the set T in type D_6 with $s = 6$ (rewritten with respect to the roots in type E_7) we have added the negative root $-\alpha_1$.

For every $\gamma \in S \setminus \{\beta_1\}$, we take the same Heisenberg set Γ_γ (rewritten with respect to the roots in type E_7) as in type D_6 with $s = 6$ and we add the maximal Heisenberg set H_{β_1} . Observe that if $\alpha \in H_{\beta_1}$ and $\beta \in \Gamma_\gamma$ with $\gamma \in S \setminus \{\beta_1\}$ then one has that $\alpha + \beta \notin S$.

Hence, by the extremal case in type D_6 (see the remark in 6.4), it follows that all conditions of lemma 3.1 hold for $y = \sum_{\gamma \in S} x_\gamma$. Then defining $h \in \mathfrak{h}_\Lambda$ by $\gamma(h) = -1$ for all $\gamma \in S$, one obtains that (h, y) is an adapted pair for \mathfrak{p} .

Finally we show that $Y(\mathfrak{p})$ is polynomial. For this we need to calculate the $\langle \mathbf{ij} \rangle$ -orbits in π and the lower and improved upper bounds for $Y(\mathfrak{p})$. The orbits are the $\Gamma_1 = \{\alpha_1\}, \Gamma_2 = \{\alpha_3\}, \Gamma_3 = \{\alpha_2, \alpha_7\}, \Gamma_4 = \{\alpha_4, \alpha_6\}$ and $\Gamma_5 = \{\alpha_5\}$. For the lower bound, we need to compute δ_Γ for all orbit Γ .

Let $\{\varepsilon_i\}_{1 \leq i \leq 8}$ be an orthonormal basis of \mathbb{R}^8 according to which the simple roots of \mathfrak{g} are expanded as in [1, Planche VI].

Recall that the fundamental weights ϖ'_i , $1 \leq i \leq 7$, $i \neq 3$, are those for the Levi factor of \mathfrak{p} .

A direct computation gives :

$$\begin{aligned} \varpi'_1 &= \frac{1}{4}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4 - \varepsilon_5 - \varepsilon_6 - \varepsilon_7 + \varepsilon_8) \text{ and } \varpi'_1 - \varpi_1 = -\frac{1}{2}\varpi_3, \\ \varpi'_2 &= \frac{1}{6}(5\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6) \text{ and } \varpi'_2 - \varpi_2 = -\frac{2}{3}\varpi_3, \\ \varpi'_4 &= \frac{1}{3}(2\varepsilon_1 - 2\varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6) \text{ and } \varpi'_4 - \varpi_4 = -\frac{4}{3}\varpi_3, \\ \varpi'_5 &= \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6) \text{ and } \varpi'_5 - \varpi_5 = -\varpi_3, \\ \varpi'_6 &= \frac{1}{3}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4 + 2\varepsilon_5 + 2\varepsilon_6) \text{ and } \varpi'_6 - \varpi_6 = -\frac{2}{3}\varpi_3, \\ \varpi'_7 &= \frac{1}{6}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4 - \varepsilon_5 + 5\varepsilon_6) \text{ and } \varpi'_7 - \varpi_7 = -\frac{1}{3}\varpi_3. \end{aligned}$$

Thus we get (recall 2.7) : $\delta_{\Gamma_1} = -2(\varpi_1 - \varpi'_1) = -\varpi_3$. Similarly one gets $\delta_{\Gamma_2} = \delta_{\Gamma_3} = \delta_{\Gamma_5} = -2\varpi_3$ and $\delta_{\Gamma_4} = -4\varpi_3$. Hence the lower bound is $(1 - e^{-\varpi_3})^{-1}(1 - e^{-2\varpi_3})^{-3}(1 - e^{-4\varpi_3})^{-1}$.

Now for the improved upper bound, for each $\gamma \in T$ we will find $t(\gamma) \in \mathbb{Q}S$ such that $\gamma + t(\gamma)$ is a multiple of ϖ_3 . Denote by s_i the i -th element of S as it is written above.

For $\gamma = -\alpha_1$, we have that $t(\gamma) = 2s_1$ and $\gamma + t(\gamma) = \varpi_3$.

For $\gamma = \alpha_4 + \alpha_5$, we have $t(\gamma) = 6s_1 + 3(s_2 + s_3) + 2(s_4 + s_5) + s_6$ and $\gamma + t(\gamma) = 4\varpi_3$.

For $\gamma = \alpha_3 + \alpha_4$, we have $t(\gamma) = 3s_1 + s_2 + s_3$ and $\gamma + t(\gamma) = 2\varpi_3$.

For $\gamma = -(\alpha_2 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7)$, we have $t(\gamma) = 3s_1 + 2s_2 + s_3 + s_5$ and $\gamma + t(\gamma) = 2\varpi_3$.

Finally, for $\gamma = -\alpha_6$, we have $t(\gamma) = 3s_1 + 2s_2 + s_3 + s_4 + s_5 + s_6$ and $\gamma + t(\gamma) = 2\varpi_3$.

We deduce that the lower bound coincides with the improved upper bound. Thus $Y(\mathfrak{p})$ is a polynomial algebra over k .

Then one checks that $h = -\alpha_1^\vee - \frac{13}{2}\alpha_2^\vee + 3\alpha_4^\vee + \frac{11}{2}\alpha_5^\vee - 2\alpha_6^\vee - \frac{1}{2}\alpha_7^\vee$. The eigenvalues of $\text{ad } h$ on the elements of \mathfrak{g}_T are respectively: 2, 17, 5, 7, 9, hence the degrees of a set of homogeneous generators of $Y(\mathfrak{p})$ are 3, 6, 8, 10, 18.

7. TYPE E_6 .

In type E_6 we know that the Poisson centre of the truncated maximal parabolic subalgebra associated to $\pi' = \pi \setminus \{\alpha_s\}$ is polynomial for $s = 3, 4, 5$ [5], resp. for $s = 2$ [21] and an adapted pair was constructed in [8], resp. in [16]. It remains to examine the cases $s = 1, 6$, and by symmetry we may just assume that $s = 6$. Recall that the numbering follows [1, Planche V].

We used computer calculations in GAP [9] and found 3,662 adapted pairs. Here we give one as close as possible to the strongly orthogonal positive roots of the Kostant cascade for Δ^+ and $\Delta_{\pi'}^+ = -\Delta_{\pi'}^-$. We choose for S the set $S = \{\beta_1, \beta_2, \beta_3, -\beta'_1, -\beta'_2 + \alpha_2\}$ or in terms of simple roots, by writing as $(a_1, a_2, a_3, a_4, a_5, a_6)$ the root $\sum_{i=1}^6 a_i \alpha_i$ our chosen set S is the set

$$S = \{(1, 2, 2, 3, 2, 1), (1, 0, 1, 1, 1, 1), (0, 0, 1, 1, 1, 0), \\ (-1, -1, -2, -2, -1, 0), (0, 0, 0, -1, -1, 0)\}.$$

Then again with computer calculations we find $T = \{\alpha_4, \alpha_6, \alpha_2 + \alpha_3 + \alpha_4\}$.

The $\langle \mathbf{ij} \rangle$ -orbits in π are $\Gamma_1 := \{\alpha_1, \alpha_6\}$, $\Gamma_2 := \{\alpha_2, \alpha_3, \alpha_5\}$ and $\Gamma_3 := \{\alpha_4\}$.

Note that π' is of type D_5 but we need to pay attention at the numbering of simple roots, which is different from the usual for D_5 . Denote by $\{\varepsilon_i\}_{i=1}^8$ an orthonormal basis of \mathbb{R}^8 according to which the roots of E_6 are expanded as in [1, Planche V]. Recall that the fundamental weights ϖ'_i , $i \in \{1, \dots, 5\}$, are those of the Levi factor of \mathfrak{p} . We have: $\varpi'_1 = \frac{1}{2}(\varepsilon_8 - \varepsilon_7 - \varepsilon_5 - \varepsilon_6)$ and $\varpi'_1 - \varpi_1 = -\frac{1}{2}\varpi_6$,

$$\varpi'_2 = \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4) - \frac{1}{4}(\varepsilon_5 + \varepsilon_6 + \varepsilon_7 + \varepsilon_8) \text{ and } \varpi'_2 - \varpi_2 = -\frac{3}{4}\varpi_6,$$

$$\varpi'_3 = \frac{1}{2}(-\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 - \varepsilon_5 - \varepsilon_6 - \varepsilon_7 + \varepsilon_8) \text{ and } \varpi'_3 - \varpi_3 = -\varpi_6,$$

$$\varpi'_4 = \varepsilon_3 + \varepsilon_4 - \frac{1}{2}(\varepsilon_5 + \varepsilon_6 + \varepsilon_7 - \varepsilon_8) \text{ and } \varpi'_4 - \varpi_4 = -\frac{3}{2}\varpi_6,$$

$$\varpi'_5 = \varepsilon_4 - \frac{1}{4}(\varepsilon_5 + \varepsilon_6 + \varepsilon_7 - \varepsilon_8) \text{ and } \varpi'_5 - \varpi_5 = -\frac{5}{4}\varpi_6.$$

We may compute δ_Γ , for each orbit Γ . We have $\delta_{\Gamma_1} = -2(\varpi_1 + \varpi_6 - \varpi'_1) = -3\varpi_6$,

$$\delta_{\Gamma_2} = -2(\varpi_2 + \varpi_3 + \varpi_5 - \varpi'_2 - \varpi'_3 - \varpi'_5) = -6\varpi_6,$$

$$\text{and } \delta_{\Gamma_3} = -2(\varpi_4 - \varpi'_4) = -3\varpi_6.$$

Hence the lower bound for $\text{ch } Y(\mathfrak{p})$ is equal to $(1 - e^{-3\varpi_6})^{-2}(1 - e^{-6\varpi_6})^{-1} \leq \text{ch } Y(\mathfrak{p})$.

We now compute the improved upper bound; recall that for every $\gamma \in T$ we need to compute the unique element $t(\gamma) \in \mathbb{Q}S$ such that $\gamma + t(\gamma)$ is a multiple of ϖ_6 .

For $\gamma = \alpha_4$, one has $t(\gamma) = 5\beta_1 + 3\beta_2 + 3\beta_3 + 2(-\beta'_2 + \alpha_2) + 4(-\beta'_1)$ and $\gamma + t(\gamma) = 6\varpi_6$.

For $\gamma = \alpha_6$, one has $t(\gamma) = 2\beta_1 + \beta_2 + \beta_3 + (-\beta'_1)$ and $\gamma + t(\gamma) = 3\varpi_6$. For $\gamma = \alpha_2 + \alpha_3 + \alpha_4$, one has $t(\gamma) = 2\beta_1 + 2\beta_2 + \beta_3 + 2(-\beta'_1)$ and $\gamma + t(\gamma) = 3\varpi_6$.

Hence the improved upper bound coincides with the lower bound and $Y(\mathfrak{p})$ is a polynomial algebra over k . Note that the element $h \in \mathfrak{h}_\Lambda$ such that $\gamma(h) = -1$ for all $\gamma \in S$ is $h = -2\alpha_1^\vee - \alpha_2^\vee + \alpha_3^\vee + 6\alpha_4^\vee - 5\alpha_5^\vee$. Then the eigenvalues of $\text{ad } h$ on the elements of \mathfrak{g}_T are 17, 5 and 7, hence the degrees of a set of homogeneous generators for $Y(\mathfrak{p})$ are 6, 8, 18.

REFERENCES

- [1] N. Bourbaki, *Groupes et algèbres de Lie*, Chapitres IV-VI, Hermann, Paris, 1968.
- [2] J. Dixmier, *Sur le centre de l'algèbre enveloppante d'une algèbre de Lie*, C.R. Acad. Sc. Paris, **265** (1967) p. 408-410.
- [3] J. Dixmier, *Algèbres enveloppantes*, Editions Jacques Gabay, les grands classiques Gauthier-Villars, Paris/Bruxelles/Montréal, 1974.
- [4] F. Fauquant-Millet, *Sur la polynomialité de certaines algèbres d'invariants d'algèbres de Lie*, Mémoire d'Habilitation à Diriger des Recherches, <http://tel.archives-ouvertes.fr/tel-00994655>.
- [5] F. Fauquant-Millet and A. Joseph, *Semi-centre de l'algèbre enveloppante d'une sous-algèbre parabolique d'une algèbre de Lie semi-simple*, Ann. Sci. École. Norm. Sup. (4) **38** (2005) no.2, 155-191.
- [6] F. Fauquant-Millet and A. Joseph, *La somme des faux degrés - un mystère en théorie des invariants*, Advances in Maths **217** (2008), 1476-1520.
- [7] F. Fauquant-Millet and A. Joseph, *Adapted pairs and Weierstrass sections*, <http://arxiv.org/abs/1503.02523>.
- [8] F. Fauquant-Millet and P. Lamprou, *Slices for maximal parabolic subalgebras of a semisimple Lie algebra*, Transformation Groups (2016), DOI : 10.1007/S00031-016-9366-9.
- [9] The GAP Group, GAP - Groups, Algorithms, Programming, version 4.8.6; 2016 (<http://www.gap-system.org>).
- [10] I. Heckenberger, *On the semi-centre of $U(\mathfrak{p})$ for parabolic subalgebras \mathfrak{p} of \mathfrak{so}_7 and \mathfrak{so}_9* , unpublished notes.
- [11] A. Joseph, *A preparation theorem for the prime spectrum of a semisimple Lie algebra*, J. Algebra **48** (1977), 241-289.
- [12] A. Joseph, *On semi - invariants and index for biparabolic (seaweed) algebras I*, J. Algebra **305** (2006), no. 1, 487-515.
- [13] A. Joseph, *On semi - invariants and index for biparabolic (seaweed) algebras II*, J. Algebra **312** (2007), no. 1, 158-193.
- [14] A. Joseph and P. Lamprou, *Maximal Poisson commutative subalgebras for truncated parabolic subalgebras of maximal index in \mathfrak{sl}_n* , Transform. Groups **12** (2007), no. 3, 549-571.
- [15] A. Joseph, *Slices for biparabolic coadjoint actions in type A*, J. Algebra **319** (2008), no. 12, 5060-5100.
- [16] A. Joseph, *Compatible adapted pairs and a common slice theorem for some centralizers*, Transform. Groups **13** (2008), no. 3-4, 637-669.
- [17] A. Joseph, *An algebraic slice in the coadjoint space of the Borel and the Coxeter element*, Advances in Maths **227** (2011), 522-585.

- [18] A. Joseph and D. Shafrir, *Polynomiality of invariants, unimodularity and adapted pairs*, Transform. Groups **15** (2010), no. 4, 851-882.
- [19] B. Kostant, *Lie group representations on polynomial rings*, Amer. J. Math. **85** (1963), 327-404.
- [20] A. Ooms, *The polynomiality of the Poisson center and semi-center of a Lie algebra and Dixmier's fourth problem*, to appear in Journal of Algebra.
- [21] D. Panyushev, A. Premet and O. Yakimova, *On symmetric invariants of centralisers in reductive Lie algebras* J. Algebra **313** (2007), no. 1, 343-391.
- [22] D. Panyushev, O. Yakimova *A remarkable contraction of semisimple Lie algebras*, Ann. Inst. Fourier (Grenoble) **62** (2012), no. 6, 2053-2068 (2013).
- [23] D. Panyushev, O. Yakimova *Parabolic contractions of semisimple Lie algebras and their invariants*, Selecta Math. (N.S.) **19** (2013), no. 3, 699-717.
- [24] D. Panyushev, O. Yakimova, *Symmetric invariants related to representations of exceptional simple groups*, arXiv : 1609.01914.
- [25] P. Tauvel and R.W.T Yu, *Affine Slice for the Coadjoint Action of a Class of Biparabolic Subalgebras of a Semisimple Lie Algebra*, Algebr. Represent. Theor. DOI : 10.1007/s10468-012-9335-5.
- [26] O. Yakimova, *A counterexample to Premet's and Joseph's conjectures*, Bull. Lond. Math. Soc. **39** (2007), no. 5, 749-754.
- [27] O. Yakimova, *Symmetric invariants of \mathbb{Z}_2 -contractions and other semi-direct products*, Int. Math. Res. Notices (2016), DOI : 10.1093/imrn/rnv381.
- [28] O. Yakimova, *Some semi-direct products with free algebras of symmetric invariants*, Proceedings of "Perspectives in Lie Theory", to appear.